

APLIKÁCIA METÓDY MINIMALIZÁCIE MAXIMÁLNEJ CHYBY V TRANSFORMÁCII POLYNÓMICKOU APROXIMÁCIOU

APPLICATION OF THE MINIMALIZATION OF MAXIMAL ERROR METHOD IN A TRANSFORMATION BY POLYNOMIAL APPROXIMATION

Metóda najmenších štvorcov je spravidla používaná na odhad koeficientov polynómicko transformácie. Článok popisuje racionalizáciu polynómicko transformácie metódou minimalizácie maximálnej chyby (Čebyševova aproximácia).

The least square method is generally used for an estimation of the coefficients of polynomial transformation. The paper describes the rationalization of polynomial approximation by the minimization of maximal error model (Chebyshev approximation).

1. Úvod - metóda minimalizácie maximálnej chyby

Pri transformácii medzi dvoma súradnicovými systémami v geodézii sa na odhad transformačných koeficientov (napr. polynómicko transformácie) spravidla používa metóda najmenších štvorcov, ktorá spočíva v minimalizácii sumy opráv na diskretné množine tzv. identických bodov, ktorých poloha je definovaná v oboch súradnicových systémoch. Ďalšou možnosťou je metóda minimalizácie maximálnej chyby, ktorá vyhovuje podmienke

$$\max_{a \leq x \leq b} |f(x) - R_{m,k}(x)| = \min, \quad (1)$$

kde

$$R_{s,t}(x) = \frac{P_s(x)}{Q_t(x)} = \frac{\sum_{j=0}^s c_j x^j}{\sum_{j=0}^t d_j x^j}, \quad d_0 = 1 \quad (2)$$

je Čebyševova racionálna aproximácia [1] všeobecnej funkcie $f(x)$ definovanej v tomto prípade diskretnou množinou identických bodov na intervale (a, b) .

2. Praktická aplikácia

Pre 78 identických bodov ($n = 78$) na území Slovenska, ktorých poloha v systémoch S-1942/83 a ETRS-89 je daná geodetickými zemepisnými súradnicami B', L' a B, L , sú určené koeficienty $a_{0,0}, a_{1,0}, \dots, a_{0,2}; b_{0,0}, b_{1,0}, \dots, b_{0,2}$ polynómicko transformácie

$$dB_i = \sum_{k=0}^2 \sum_{l=0}^k a_{k-l,l} \Delta B_i^{k-l} \Delta L_i^l = a_{0,0} + a_{1,0} \Delta B_i + a_{0,1} \Delta L_i + a_{1,1} \Delta B_i \Delta L_i + a_{2,0} \Delta B_i^2 + a_{0,2} \Delta L_i^2,$$

1. Introduction - the minimization of maximal error method

To estimate the coefficients of transformation (e.g. the polynomial transformation) in the transformation between two coordinate systems in geodesy the least square method is generally used. This method resides in the minimization of the sum of corrections on a discrete set, which contains the identical points, whose position is defined in both coordinate systems. Another possibility is the minimization of maximal error method, which corresponds with the condition

$$\max_{a \leq x \leq b} |f(x) - R_{m,k}(x)| = \min, \quad (1)$$

where

$$R_{s,t}(x) = \frac{P_s(x)}{Q_t(x)} = \frac{\sum_{j=0}^s c_j x^j}{\sum_{j=0}^t d_j x^j}, \quad d_0 = 1 \quad (2)$$

is the rational Chebyshev approximation [1] of the common function $f(x)$ defined in this case by the discrete set of identical points on the interval (a, b) .

2. Practical application

For 78 identical points ($n = 78$) in the territory of Slovakia, whose position in the systems S-1942/3 and ETRS-89 is given by geodetic coordinates B', L' and B, L , the coefficients $a_{0,0}, a_{1,0}, \dots, a_{0,2}; b_{0,0}, b_{1,0}, \dots, b_{0,2}$ of polynomial approximation are estimated

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$$dL_i = \sum_{k=0}^2 \sum_{l=0}^k a_{k-l,l} \Delta B_i^{k-l} \Delta L_i^l = b_{0,0} + b_{1,0} \Delta B_i + b_{0,1} \Delta L_i + a_{1,1} \Delta B_i \Delta L_i + b_{2,0} \Delta B_i^2 + b_{0,2} \Delta L_i^2, \quad (3)$$

kde $\Delta B_i = B_i - n^{-1} \sum_{i=1}^n B_i$, $\Delta L_i = L_i - n^{-1} \sum_{i=1}^n L_i$, $dB_i = (B'_i - B_i)M_i$, $dL_i = (L'_i - L_i)N_i \cos B_i$,
 M je meridiánový polomer krivosti a N je priečný polomer krivosti, M is meridian radius of curvature and N is prime vertical radius of curvature, using the least square method in the system of equations

$$A_{dB} \delta x_{dB}^{(0)} - l_{dB} = v_{dB}^{(0)}, \quad A_{dL} \delta x_{dL}^{(0)} - l_{dL} = v_{dL}^{(0)},$$

$$\delta x_{dB}^{(0)} = (A_{dB}^T P_{dB} A_{dB})^{-1} A_{dB}^T P_{dB} l_{dB}, \quad \delta x_{dL}^{(0)} = (A_{dL}^T P_{dL} A_{dL})^{-1} A_{dL}^T P_{dL} l_{dL},$$

$$AdB \equiv AdL = \begin{pmatrix} 1 & \Delta B_1 & \Delta L_1 & \Delta B_1 \Delta L_1 & \Delta B_1^2 & \Delta L_1^2 \\ 1 & \Delta B_2 & \Delta L_2 & \Delta B_2 \Delta L_2 & \Delta B_2^2 & \Delta L_2^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \Delta B_n & \Delta L_n & \Delta B_n \Delta L_n & \Delta B_n^2 & \Delta L_n^2 \end{pmatrix}$$

$$(\delta x_{dB}^{(0)}) = (a_{0,0}^{(0)} \ a_{1,0}^{(0)} \ a_{0,1}^{(0)} \ a_{1,1}^{(0)} \ a_{2,0}^{(0)} \ a_{0,2}^{(0)}), \quad (\delta x_{dL}^{(0)}) = (b_{0,0}^{(0)} \ b_{1,0}^{(0)} \ b_{0,1}^{(0)} \ b_{1,1}^{(0)} \ b_{2,0}^{(0)} \ b_{0,2}^{(0)}), \quad (4)$$

$$I_{dB} = \begin{pmatrix} dB_1 \\ dB_2 \\ \vdots \\ dB_n \end{pmatrix} = \begin{pmatrix} (B'_1 - B_1)M_1 \\ (B'_2 - B_2)M_2 \\ \vdots \\ (B'_n - B_n)M_n \end{pmatrix}, \quad I_{dL} = \begin{pmatrix} dL_1 \\ dL_2 \\ \vdots \\ dL_n \end{pmatrix} = \begin{pmatrix} (L'_1 - L_1)N_1 \cos B_1 \\ (L'_2 - L_2)N_2 \cos B_2 \\ \vdots \\ (L'_n - L_n)N_n \cos B_n \end{pmatrix},$$

$$P_{dB} = \begin{pmatrix} P_{dB_1} & 0 & \dots & 0 \\ 0 & P_{dB_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{dB_n} \end{pmatrix}, \quad P_{dL} = \begin{pmatrix} P_{dL_1} & 0 & \dots & 0 \\ 0 & P_{dL_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{dL_n} \end{pmatrix},$$

za podmienky minimalizácie sumy opráv na diskretné množine bodov $i = 1, 2, \dots, n$,

$$\sum_{i=1}^n v_{dB,i}^{(0)} P_{dB,i} v_{dB,i}^{(0)} = v_{dB}^{(0)T} P_{dB} v_{dB}^{(0)} = \min., \quad \sum_{i=1}^n v_{dL,i}^{(0)} P_{dL,i} v_{dL,i}^{(0)} = v_{dL}^{(0)T} P_{dL} v_{dL}^{(0)} = \min. \quad (5)$$

na intervale $(-1, 1)$. Váhy sú funkciou stredných súradnicových chýb na identických bodoch

$$P_{dB_i} = f(m_{dB_i}, m_{dB_i}), \quad P_{dL_i} = f(m_{dL_i}, m_{dL_i}).$$

Na obr. 1 a obr. 2 sú znázornené vektory opráv $v^{(0)}$ a veľkosť zložiek $v_{dB}^{(0)}$, $v_{dL}^{(0)}$ vektorov opráv polynomickej transformácie.

Podľa druhého Remesovho algoritmu je možné určiť koeficienty Čebyševovej aproximácie riešením systému rovníc postupnými iteráciami [2]

$$f(x_i) - R_{s,t}(x) = f(x_i) - \frac{\sum_{j=0}^s c_j(x_i)^j}{\sum_{j=0}^t d_j(x_i)^j} = (-1)^j \max |m| \Rightarrow \sum_{j=0}^s c_j(x_i)^j - [f(x_i) - (-1)^j \max |m^{(l)}|] \sum_{j=1}^t d_j(x_i)^j = f(x_i) - (-1)^j \max |m^{(l+1)}|,$$

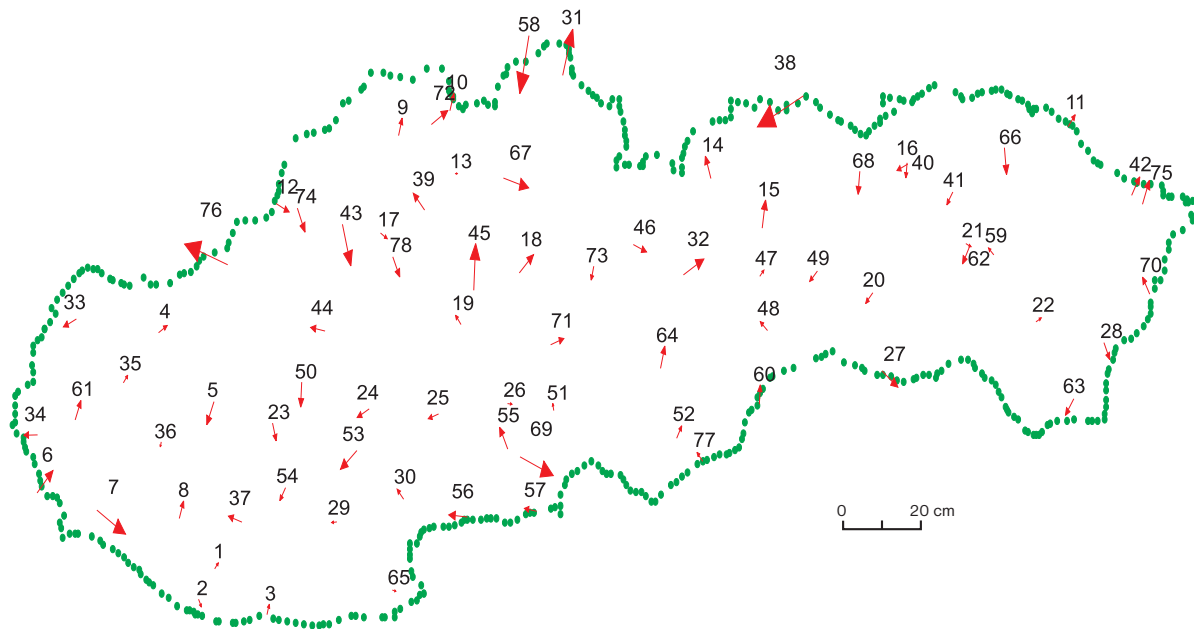
$$i = 0, 1, \dots, s + t + 1, \quad l = 0, 1, \dots, \text{pre } t \neq 0, \quad (6)$$

under the condition of minimization of sum of corrections on the discrete set of points $i = 1, 2, \dots, n$,

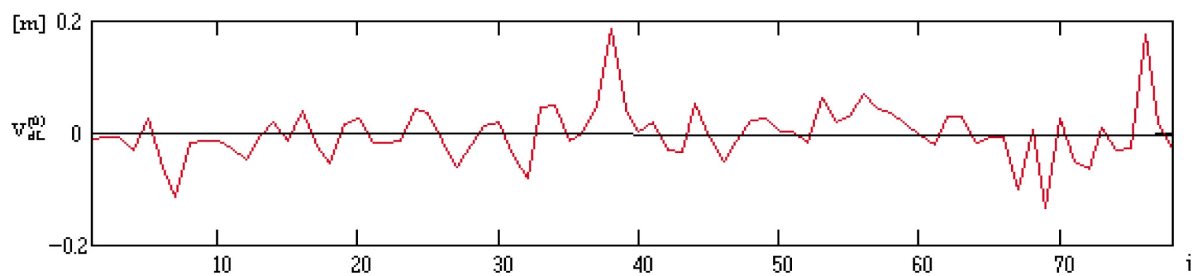
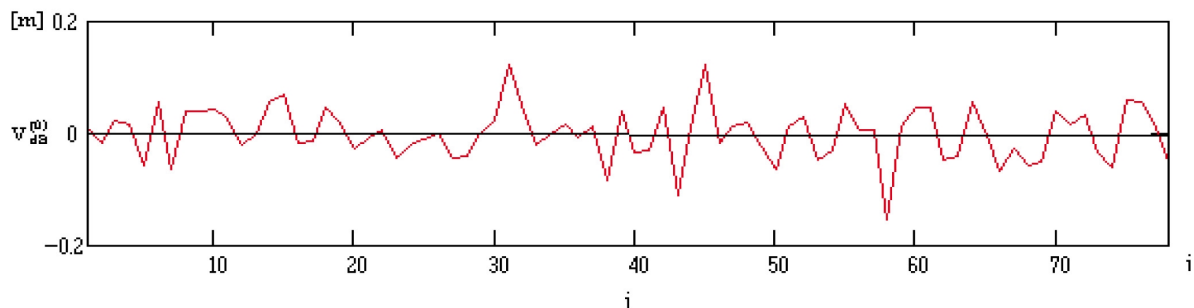
on the interval $(-1, 1)$. The weights are function of the mean coordinate errors on the identical points

Figures 1 and 2 show the vectors of corrections $v^{(0)}$ and the value of the correction vector components $v_{dB}^{(0)}$, $v_{dL}^{(0)}$ in the polynomial transformation.

According to the second Remes' algorithms it is possible to determine the coefficients of Chebyshev approximation by solving the system of equations using iterative operations consecutively [2]



Obr. 1. Vektory opráv $v^{(0)}$ polynomickej aproximácie
Fig. 1. Vectors of corrections $v^{(0)}$ of polynomial transformation



Obr. 2. Veľkosť zložiek opráv $v_{dB}^{(0)}$, $v_{dL}^{(0)}$ polynomickej aproximácie
Fig. 2. Value of correction vector components $v_{dB}^{(0)}$, $v_{dL}^{(0)}$ of polynomial transformation

a pre $t = 0$ je systém lineárnych rovníc v tvare

$$f(x_i) - \sum_{j=0}^s c_j(x_i)^j = (-1)^i m. \quad (7)$$

Pri riešení systému lineárnych rovníc (7) pre prípad koeficientov polynomickej transformácie platí

and for $t = 0$ a linear system of equations has a form

$$f(x_i) - \sum_{j=0}^s c_j(x_i)^j = (-1)^i m. \quad (7)$$

In solving the linear system of equations (7), for the coefficients of polynomial transformation applies

$$f_{dB}(x) - \delta_i \max |v_{dB}^{(0)}| = \begin{pmatrix} f_{dB}(x_1) \\ f_{dB}(x_2) \\ \vdots \\ f_{dB}(x_n) \end{pmatrix} - \delta_i \max |v_{dB}^{(0)}| \equiv I_{dB} - \delta_i \max |v_{dB}^{(0)}| = \begin{pmatrix} (B'_1 - B_1)M_1 \\ (B'_2 - B_2)M_2 \\ \vdots \\ (B'_n - B_n)M_n \end{pmatrix} - \delta_i \max |v_{dB}^{(0)}|,$$

$$f_{dL}(x) - \delta_i \max |v_{dL}^{(0)}| = \begin{pmatrix} f_{dL}(x_1) \\ f_{dL}(x_2) \\ \vdots \\ f_{dL}(x_n) \end{pmatrix} - \delta_i \max |v_{dL}^{(0)}| \equiv I_{dL} - \delta_i \max |v_{dL}^{(0)}| = \begin{pmatrix} (L'_1 - L_1)N_1 \cos B_1 \\ (L'_2 - L_2)N_2 \cos B_2 \\ \vdots \\ (L'_n - L_n)N_n \cos B_n \end{pmatrix} - \delta_i \max |v_{dL}^{(0)}|, \quad (8)$$

$$AdB \equiv AdL = \begin{pmatrix} 1 & \Delta B_1 & \Delta L_1 & \Delta B_1 \Delta L_1 & \Delta B_1^2 & \Delta L_1^2 & \delta_1 \\ 1 & \Delta B_2 & \Delta L_2 & \Delta B_2 \Delta L_2 & \Delta B_2^2 & \Delta L_2^2 & \delta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \Delta B_n & \Delta L_n & \Delta B_n \Delta L_n & \Delta B_n^2 & \Delta L_n^2 & \delta_n \end{pmatrix}$$

$\delta_i = 1$ pre $v_i \leq 0$ a $\delta_i = -1$ pre $v_i > 0$,

$$(\delta x_{dB}^{(1)})^T = (a_{0,0}^{(1)} \ a_{1,0}^{(1)} \ a_{0,1}^{(1)} \ a_{1,1}^{(1)} \ a_{2,0}^{(1)} \ a_{0,2}^{(1)} \ \max |v_{dB}^{(1)}|), \quad (\delta x_{dL}^{(1)})^T = (b_{0,0}^{(1)} \ b_{1,0}^{(1)} \ b_{0,1}^{(1)} \ b_{1,1}^{(1)} \ b_{2,0}^{(1)} \ b_{0,2}^{(1)} \ \max |v_{dL}^{(1)}|),$$

kde $\max |v_{dB}^{(1)}|$, $\max |v_{dL}^{(1)}|$ sú maximálne zložky opráv l -tej iterácie, $l = 0, I, \dots$

Koeficienty Čebyševovho aproximačného polynómu boli určené podľa rovníc

where $\max |v_{dB}^{(1)}|$, $\max |v_{dL}^{(1)}|$ are the maximal components of correction l -iteration, $l = 0, I, \dots$

The coefficients of Chebyshev approximation polynomial are determined using following equations

$$\delta x_{dB}^{(l)} = (A_{dB}^T P_{dB} A_{dB})^{-1} A_{dB}^T P_{dB} [f_{dB}(x) - \delta_i \max |v_{dB}^{(0)}|] = (A_{dB}^T P_{dB} A_{dB})^{-1} A_{dB}^T P_{dB} [I_{dB} - \delta_i \max |v_{dB}^{(0)}|],$$

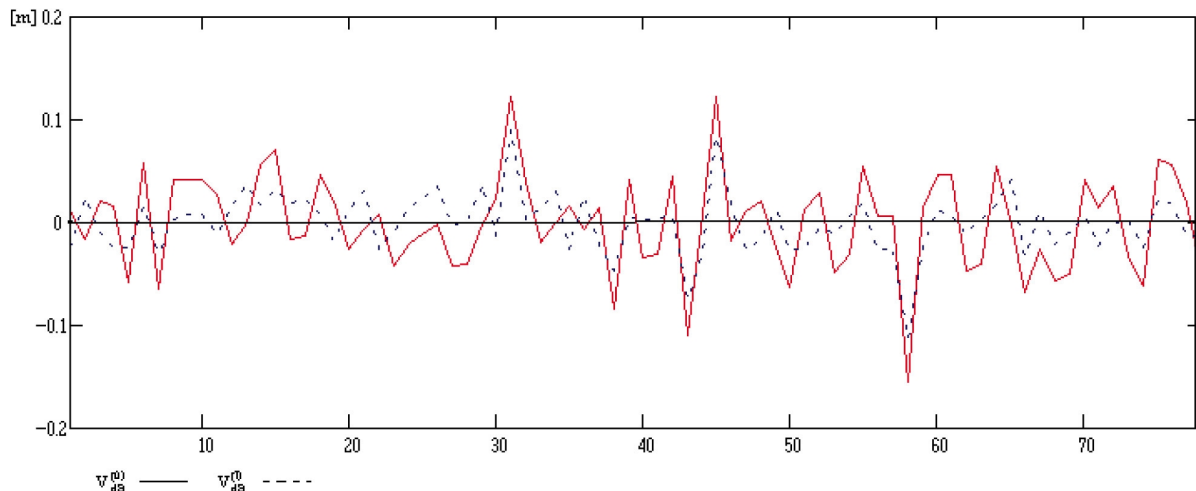
$$\delta x_{dL}^{(l)} = (A_{dL}^T P_{dL} A_{dL})^{-1} A_{dL}^T P_{dL} [f_{dL}(x) - \delta_i \max |v_{dL}^{(0)}|] = (A_{dL}^T P_{dL} A_{dL})^{-1} A_{dL}^T P_{dL} [I_{dL} - \delta_i \max |v_{dL}^{(0)}|], \quad (9)$$

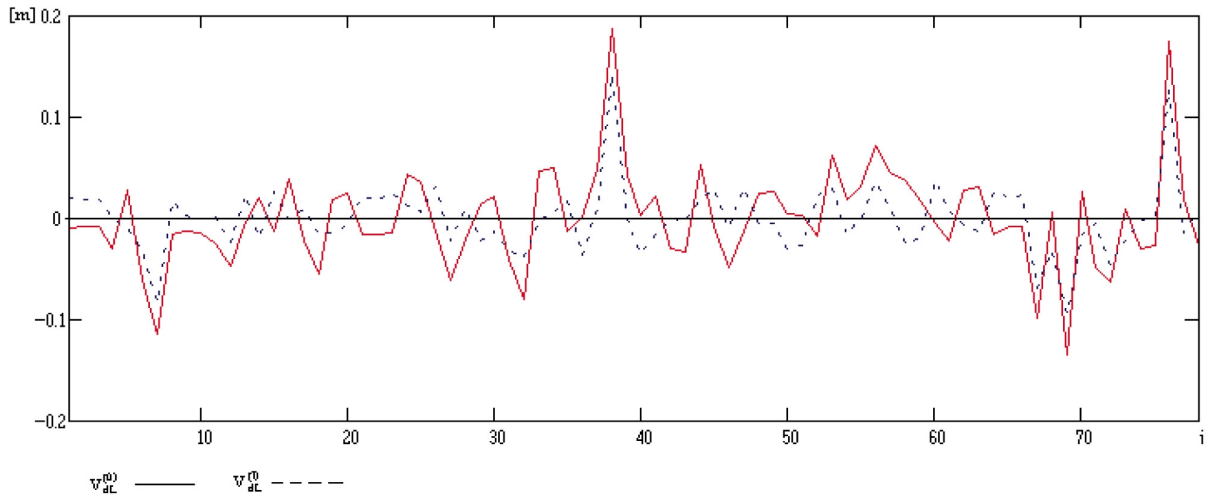
a pre vektory opráv platí

and for correction vectors applies:

$$A_{dB} \delta x_{dB}^{(l)} - [f_{dB}(x) - \delta_i \max |v_{dB}^{(0)}|] = A_{dB} \delta x_{dB}^{(l)} - [I_{dB} - \delta_i \max |v_{dB}^{(0)}|] = v_{dB}^{(l)},$$

$$A_{dL} \delta x_{dL}^{(l)} - [f_{dL}(x) - \delta_i \max |v_{dL}^{(0)}|] = A_{dL} \delta x_{dL}^{(l)} - [I_{dL} - \delta_i \max |v_{dL}^{(0)}|] = v_{dL}^{(l)}, \quad (10)$$





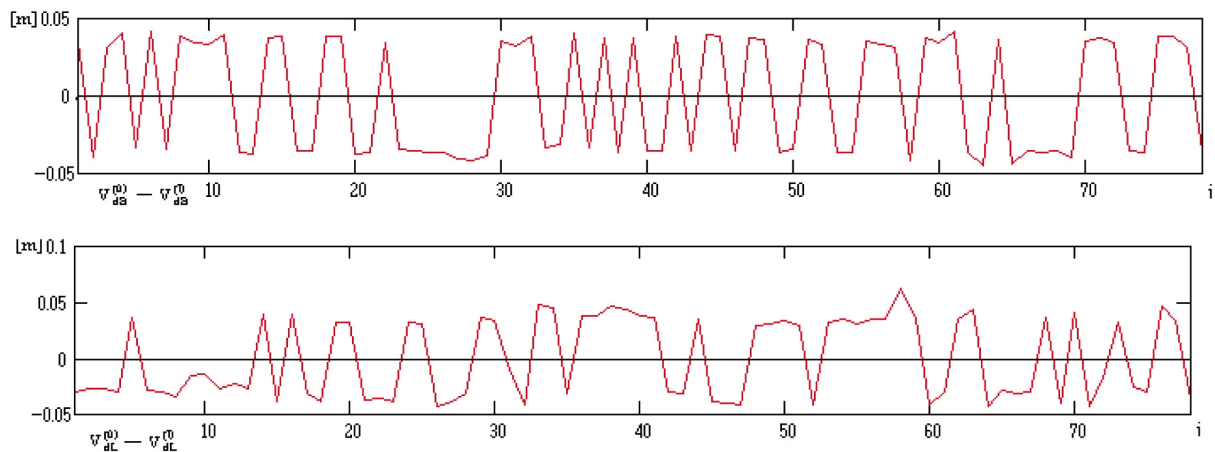
Obr. 3. Zložky opráv $v_{dB}^{(l)}, v_{dL}^{(l)}$ Čebyševovej aproximácie a zložky opráv $v_{dB}^{(0)}, v_{dL}^{(0)}$ polynómnickej aproximácie
Fig.3 Correction components of Chebyshev approximation $v_{dB}^{(l)}, v_{dL}^{(l)}$ and the correction components of polynomial approximation $v_{dB}^{(0)}, v_{dL}^{(0)}$

Pri $t = 0$ prvá iterácia postačuje.

For $t = 0$, the first iteration is sufficient.

Na obr. 3 a obr. 4 sú porovnané veľkosti zložiek opráv Čebyševovej $v_{dB}^{(l)}, v_{dL}^{(l)}$ a polynómnickej aproximácie $v_{dB}^{(0)}, v_{dL}^{(0)}$ a na obr. 5 sú vektory opráv $v^{(l)}$ Čebyševovej aproximácie.

Figures 3 and 4 show a comparison between the value of the correction components of Chebyshev approximation $v_{dB}^{(l)}, v_{dL}^{(l)}$ and polynomial approximation $v_{dB}^{(0)}, v_{dL}^{(0)}$. Figure 5 shows the correction vectors $v^{(l)}$ of Chebyshev approximation.



Obr. 4. Rozdiely zložiek opráv Čebyševovej a polynómnickej aproximácie
Fig. 4. Differences between the correction components of Chebyshev and polynomial approximation

Koeficienty polynómnickej a Čebyševovej aproximácie

Tab. 1

	Koeficienty polynómnickej aproximácie	Koeficienty Čebyševovej aproximácie
$a_{0,0}$	-37,918	-37,920
$a_{1,0}$	82,708	82,480
$a_{0,1}$	95,605	95,680
$a_{1,1}$	159,467	143,090

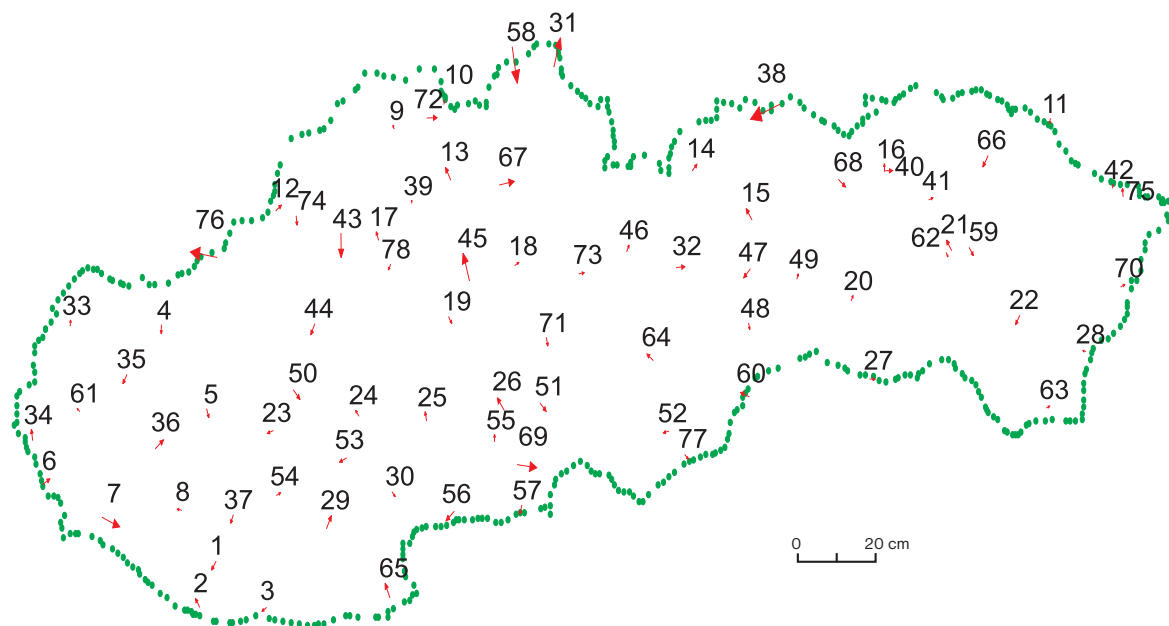
Coefficients of polynomial and Chebyshev approximation

Tab. 1

	Coefficients of polynomial approximation	Coefficients of Chebyshev approximation
$a_{0,0}$	-37.918	-37.920
$a_{1,0}$	82.708	82.480
$a_{0,1}$	95.605	95.680
$a_{1,1}$	159.467	143.090

	Koeficienty polynomickej aproximácie	Koeficienty Čebyševovej aproximácie
$a_{2,0}$	161,477	203,250
$a_{0,2}$	34,594	35,100
$\max v_{dB}^{(I)} $	-	-0,12 [m]
$b_{0,0}$	-124,851	-124,845
$b_{1,0}$	-2,281	-2,836
$b_{0,1}$	23,059	23,203
$b_{1,1}$	-32,703	-6,1319
$b_{2,0}$	-142,091	-250,014
$b_{0,2}$	18,453	10,609
$\max v_{dB}^{(I)} $	-	0,1516 [m]

	Coefficients of polynomial approximation	Coefficients of Chebyshev approximation
$a_{2,0}$	161.477	203.250
$a_{0,2}$	34.594	35.100
$\max v_{dB}^{(I)} $	-	-0.12 [m]
$b_{0,0}$	-124.851	-124.845
$b_{1,0}$	-2.281	-2.836
$b_{0,1}$	23.059	23.203
$b_{1,1}$	-32.703	-6.1319
$b_{2,0}$	-142.091	-250.014
$b_{0,2}$	18.453	10.609
$\max v_{dB}^{(I)} $	-	0.1516 [m]



Obr. 5. Vektory opráv $v^{(I)}$ Čebyševovej aproximácie
Fig. 5. Correction vectors $v^{(I)}$ of Chebyshev approximation

Jednotkové stredné chyby polynomickej a Čebyševovej aproximácie

Tab. 2

	Jednotkové stredné chyby polynomickej aproximácie	Jednotkové stredné chyby Čebyševovej aproximácie
$m_{0,dB}$	$\pm 4,9$ cm	$\pm 3,0$ cm
$m_{0,dL}$	$\pm 5,1$ cm	$\pm 5,1$ cm

Unit mean errors of polynomial and Chebyshev approximation

Tab. 2

	Unit mean errors of polynomial approximation	Unit mean errors of Chebyshev approximation
$m_{0,dB}$	± 4.9 cm	± 3.0 cm
$m_{0,dL}$	± 5.1 cm	± 5.1 cm

3. Záver

Z porovnania vektorov polynomickej a Čebyševovej aproximácie vyplýva skutočnosť, že okrem samotného zníženia maximálnej chyby aproximácie (v v našom prípade maximálnej opravy v súbore identických bodov) pri použití aproximácie Čebyševovým polynómom dochádza aj k zníženiu veľkosti opráv na prak-

3. Conclusion

The comparison between vectors of the polynomial approximation and Chebyshev approximation results in fact that not only the maximal error of approximation is reduced (in our case, the maximal correction on the set of identical points), but using approximation by Chebyshev polynomial reduction of the value of

ticky celej množine identických bodov a tým aj k zníženiu jednotkovej strednej chyby aproximácie: Čebyševova aproximácia racionalizuje transformačný polynóm.

corrections relates to whole set of identical points and also to reduction of unit mean error of approximation: *Transformation polynomial is rationalised by Chebyshev approximation.*

Literatúra - References:

- [1] RALSTON, A.: *A first course in numerical analysis*, Mc Graw - Hill Book Company, New York 1965.
- [2] RALSTON, A.: *Rational Chebyshev Approximation by Remes' Algorithmus*, Numerical Mathematica 7, 1965.