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ON A CERTAIN TRANSPORT SCHEDULING PROBLEM FOR HETEROGENEOUS BUS FLEET

In this paper we consider a certain transport scheduling problem for heterogeneous bus fleet. We suppose that some restrictions are given for sets of vehicles and trips. We study some special cases of this problem that can be solved in polynomial time.

1. Introduction

We deal with the problem of finding an admissible transport schedule for heterogeneous bus fleet. We suppose that the place of departure and arrival is the same for every trip and the set of vehicles that can perform a given trip is restricted. This problem was studied in [2] and its special case (for two types of vehicles) in [3]. Its complexity is still an open question. It is known, that the generalisation of this problem (where the places of departure and arrival can be different) is NP-hard.

Let $V = \{v_1, ..., v_m\}$ be the set of vehicles and $S = \{s_1, ..., s_n\}$ be the set of trips. Every trip $s_i = (t_i^D, t_i^A, L_i)$, $0 \leq t_i^D < t_i^A$ is determined by the time of departure t_i^D, time of arrival t_i^A and by the list of vehicles $L_i \subseteq V$ that can perform the trip s_i. We suppose that

1. the place of departure and arrival is the same for every trip,
2. $t_i^X \neq t_j^Y$ for all $i, j \in \{1, ..., n\}$, $i \neq j$ and $X, Y \in \{D, A\}$
3. $\forall s_i, s_j \in S$: $(i < j \Leftrightarrow \mathrm{t_i^D} < t_j^D)$,
4. the trip s_i precedes the trip s_j ($s_i \prec s_j$) if $i < j$ and $t_i^A < t_j^D$.

From the condition (4), it is easy to show that $(S, \prec)$ is a partially ordered set with the relation "$\prec$". More about partially ordered sets (posets) can be found in [8].

We say that the trips $\forall s_i, s_j \in S$ are of the same type if $L_i = L_j$. An admissible running board of the vehicle v_i is the chain $T_i = S_{i_1} \prec S_{i_2} \prec ... \prec S_{i_k}$ where vi $\in L_{i_1} \cap L_{i_2} \cap ... \cap L_{i_k}$.

A schedule $R = (T_1, ..., T_m)$ is admissible if T_i is admissible running board of the vehicle v_i (for $i = 1, ..., m$), $T_1 \cup T_2 \cup ... \cup T_m = S$ and $T_i \cap T_j = \emptyset$ $(i,j \in \{\forall 1, 2, ..., m\}$. A set of all admissible schedules will be denoted by $\Re$. We will deal with the decision problem, whether $\Re = \emptyset$ or $\Re \neq \emptyset$.

2. Mathematical models

In this section we present two models of the mentioned decision problem. The first is *the model based on a bivalent programming formulation*.

By ordering increasingly the times of departures and arrivals, we obtain the vector

$$\vec{t} = (t_1^D, ..., t_1^A, ..., t_i^D, ..., t_i^A) = (t_1, t_2, ... t_{2n})$$

where $t_k < t_l \Leftrightarrow k < l$. Let $K = \{1, 2, ..., 2n\}$ be the set of the indices of the times in the vector $\vec{t}$. Let $I = \{1, 2, ..., m\}$, be the index set of the vehicles and J = {1, 2, . . . , n} be the indices of the trips.

Let x_{ij} be a decision variable with the value $x_{ij} = 1$ if s_j is performed by v_i and $x_{ij} = 0$ otherwise. We need to decide if there exists a matrix $X = (x_{ij})m \times n$ for which the following constraints hold:

$$\sum_{i \in I} a_{ij} x_{ij} = 1 \quad j \in J, \tag{1}$$

$$\sum_{j \in J} a_{ij} b_{kj} x_{ij} \leq 1 \quad i \in I, k \in K \tag{2}$$

$$x_{ij} \in \{0, 1\} \quad i \in I, j \in J \tag{3}$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \in L_j \\ 0 & \text{if } v_i \notin L_j \end{cases} \tag{4}$$

$$b_{kj} = \begin{cases} 1 & \text{if } t_j^D \leq t_k \leq t_j^A \\ 0 & \text{if otherwise} \end{cases} \tag{5}$$

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Conditions (1) ensure that every trip s_j is assigned to exactly one vehicle (for which $v_i \in L_j$).

Conditions (2) ensure that the vehicle can perform at most one trip in every moment. The existence of some polynomial method for finding the matrix X is still an open question.

The model based on graph theory.

Definition 1. (see [6]) Let $G = (V, E)$ be a graph and $C \neq \emptyset$ be a set of colors.

A (vertex) coloring of G is a function $c : V \to C$ such that subgraphs induced by each color class have no edges.

Definition 2. (see [5]) Let the sets $Lv \subseteq C$ be assigned to each vertex $v \in V$. We call the vertex coloring $c : V \to C$ of G with property $\forall v \in V \; c(v) \in L_v$ a coloring from the lists (or list coloring).

Definition 3. (see [1]) A conditional coloring of G with respect to a graphical property $\mathfrak{P}$ is an assignment of colors to its vertices so that subgraphs induced by each color class satisfy the property $\mathfrak{P}$.

Since $(S, \prec)$ is the poset, we can define a transitive digraph $G_S = (V_S, E_S)$ where $V_S = S$ and $\forall s_i, s_j \in S \; s_j \; (s_i, s_j) \in E_S \Leftrightarrow s_i \prec s_j$. The problem of finding the admissible schedule can be solved as the problem of finding some conditional coloring of G_S from the lists which satisfies the property P_{op}: vertices of the same color form an oriented path (or a transitive tournament) in G_S. Every color corresponds to one vehicle. For every vertex (trip), the list of admissible colors (the set of vehicles that can perform this trip) is specified. Vertices of the same color form a chain in $(S, \prec)$ which corresponds to some admissible running board.

The existence of polynomial algorithm for conditional coloring from the lists with respect to the property that vertices of the same color form an oriented path in G_S is an unsolved problem.

3. Cases that can be solved in polynomial time

In this section we study some special cases that can be solved in polynomial time. We prefer a graph theoretical approach.

1. Let $|L_1| = \cdots = |L_n| = 1$. This case is trivial, every vertex $v_i \in G_S$ is colored by unique color from the list L_i. There is an admissible schedule if and only if the vertices of the same color form an oriented path in G_S. We are able to verify this in polynomial time.
2. Let $|L_1| = \cdots = |L_n| = 2$. Since G_S is transitive, vertices of every oriented path form also a transitive tournament in G_S. Let $G = (V, E)$ be a graph for which $V = V_S$ and the edge $\{s_i, s_j\} \in E$ if and only if the trips s_i, s_j are incomparable by relation $\prec$. It is easy to show that there is a coloring of G from the lists $L_1 = L_2 = \cdots = L_n$ if and only if there exists a conditional coloring of G_S from the lists with respect to the property P_{op}. It is known (see also [7]) that the list coloring of the graph is solvable in polynomial time if every list contains exactly two elements (colors).
3. If $L_1 = L_2 = \cdots = L_n \subseteq V$ then we are able to transform this case to the problem of covering a poset by disjoint union of chains – the solution of this problem is based on the famous Dilwort's Chain Decomposition Theorem [4] and we are able to find it in polynomial time.
4. Let $L_i \cap L_j = \emptyset$ or $L_n \subseteq V$ for all $i, j \{1, ..., n\}$. We can decompose $(S, \prec)$ into subposets $(S_1, \prec)$, ..., $(S_k, \prec)$ (see [8]) where $S_i \cap S_j = \emptyset$, $S_1 \cup \cdots \cup S_k = S$, and two trips s_i, s_j belong to the same subposet if $L_i = L_j$. We can solve this case for every poset $(S_i, \prec)$ separately as in (3).

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