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THE EIGENVALUE APPROXIMATIONS OF THE LAPLACE OPERATOR DEFINED ON A DOMAIN WITH STRONGLY DEFORMED BOUNDARY

In this paper the eigenvalue approximations of the Laplace operator defined on a domain with strongly deformed boundary are presented. Because the exact eigenfunctions exhibit complicated behaviour in the vicinity of singular points of the used conformal mapping, the B-spline trial functions are used in order to improve the quality of the eigenfunction approximations near the singular points.

1. Introduction

The eigenvalue problem for the two-dimensional Laplace operator defined on domains with complicated boundary shape arises in many practical situations, for example in mechanical engineering, microwave theory and techniques and biomechanics [3, 6]. The complicated shape form of domains is of interest in practice when the Laplace operator defined on the standard domains as a circle and square does not offer the optimum eigenvalue distribution needed for meeting the design requirements. Standard methods and their combinations with various special techniques have achieved the solution of such problems. The author of this paper [5] has recently presented the eigenvalue computations using this technique based on the sine trial functions. However, because of the presence of geometrical singularities of the exact eigenfunctions, the convergence of the Ritz eigenvalue approximations for the large deformation of domain under consideration is very slow. On the other hand, these singularities are of local character and in this case more precise approximations can be obtained using a local approximation, for example the spline approximation and finite element method.

In this paper the eigenvalue approximations of the Laplace operator defined on a domain with strongly deformed boundary are presented. Because of the presence of shape singularities of the exact eigenfunctions the *B*-spline trial functions are used in order to improve the quality of the eigenfunction approximations near the singular points.

2. Formulation of the problem

The eigenvalue problem for the Laplace operator, known also as the homogeneous Helmholtz equation, is given by

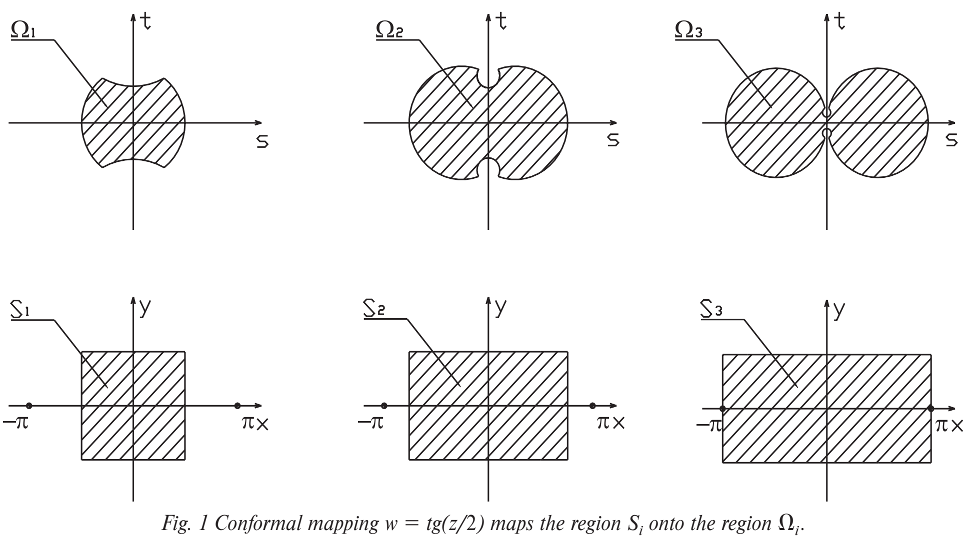

Fig. 1 Conformal mapping $w = tg(z/2)$ maps the region S_i onto the region Ω_i.

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$$-\frac{\partial^2\psi}{\partial s^2} - \frac{\partial^2\psi}{\partial t^2} = \lambda\psi \qquad \text{in } \Omega \tag{1}$$

with the Dirichlet boundary condition

$$\psi = 0 \qquad \text{on } \partial\Omega, \tag{2}$$

where Ω is a bounded two-dimensional domain with a piecewise smooth boundary.

We deal with the problem in which the domain Ω in the w – plane ($w = s+it$) is generated by conformal mapping $w = f(z)$ of a rectangle in the z – plane ($z = x + iy$). The conformal mapping $w = tg(z/2)$ maps the region S_i in the z – plane onto the region Ω_i in the w – plane bounded by arcs of the unit circle and a pair of orthogonal circles, see Fig. 1.

Using the conformal mapping $w = f(z)$ the eigenvalue problem (1), (2) is transformed to the equation

$$-\Delta U(x, y) = \sigma(x, y)\, \lambda U(x, y) \qquad \text{in} \tag{3}$$

with the Dirichlet boundary condition

$$U = 0 \qquad \text{on } \partial S. \tag{4}$$

Here the function $\sigma(x, y) = \left| \frac{df(z)}{dz} \right|$ is defined as follows

$$\sigma(x, y) = \frac{1}{(\cos x + \cosh y)^2}\,, \tag{5}$$

The nearest singular points of conformal mapping $w = tg(z/2)$ to the regions S_i are $T_1 = [-\pi, 0]$, $T_2 = [\pi, 0]$. The shapes of the function $\sigma(x, y)$ corresponding to the weakly deformed domain Ω_1 and to the strongly deformed domains Ω_3 are plotted in the left and right in Figure 2, respectively.

The domains Ω_1 and Ω_3 are created by the conformal mapping $w = tg(z/2)$ of the square $S_1 = \langle -\pi/2, \pi/2\rangle \times \langle -\pi/2, \pi/2\rangle$ and the rectangle $S_3 = \langle -1.9\pi/2, 1.9\pi/2\rangle \times \langle -\pi/2, \pi/2\rangle$, respectively.

3. Spline approximation

Definition 1. Let t_i, $i = 1, 2. ..., n$ be an increasing sequence of points of the real axis. The function $B_i^k(t)$ with $i + k \leq n$ is called i-th algebraic B-spline of order k (see Fig. 3), if the following properties are satisfied:

(a) $B_i^k(t) \neq 0$ only for $t \in (t_i, t_{i+k})$,
(b) $B_i^k(t)$ is algebraic polynomial of order $(k - 1)$ on the each interval $\langle t_i, t_{l+1})$ $i \leq l \leq i + k - 1$,
(c) $B_i^k(t)$ is continuous function with continuous derivatives up to the order $(k - 2)$ on the whole real axis.

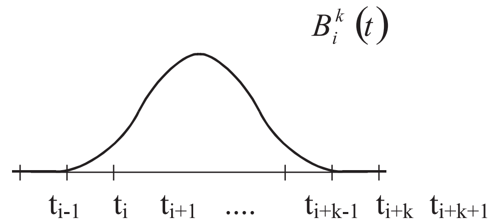

Fig. 3 Shape of the i-th algebraic B-spline of order

For the calculation of the B-splines and their derivatives the following numerically stable recurrence relations [2] are used

$$B_i^k(t) = t - t_i \backslash t_{i+k-1} - t_i\, B_i^{k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}}\, B_{i+1}^{k-1}(t) \tag{6}$$

$$\text{if } B_i^1(t) = \begin{cases} 1 & t \in \langle t_i, t_{l+1}) \\ 0 & t \notin \langle t_i, t_{l+1}) \end{cases} \tag{7}$$

$$\text{and } B_i^k(t)^{(m)} = (k-1)\left[\frac{B_i^{k-1}(t)^{(m-1)}}{t_{i+k-1} - t_i} - \frac{B_{i+1}^{k-1}(t)^{(m-1)}}{t_{i+k} - t_{i+1}} \right]. \tag{8}$$

4. Numerical experiments

The numerical experiments presented in this article are based on the Rayleigh - Ritz method applied on the equation (3) using the B-spline trial functions of the form

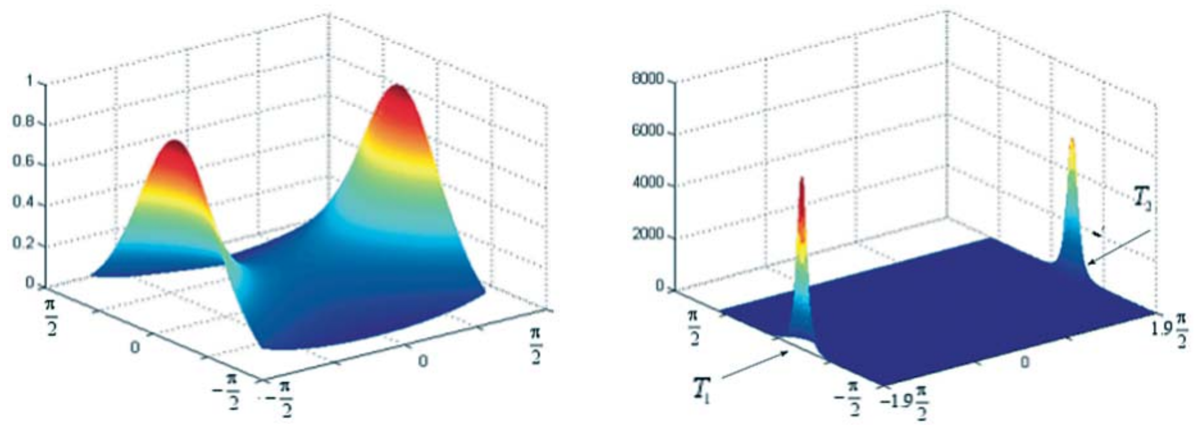

Fig. 2 Shapes of the function $\sigma(x, y)$ corresponding to the domain Ω_1 (left) and Ω_3 (right)

$$\psi_{k,l}(x, y) = (\cos x + \cosh y)\, \tilde{\psi}_{k,l}(x, y), \tag{9}$$

where

$$\tilde{\psi}_{k,l}(x, y) = (x - a)(b - x)\left(y + \frac{\pi}{2}\right)\left(\frac{\pi}{2} - y\right) B_k^r(x)\, B_l^r(y) \tag{10}$$

for $r = 8$, $k = 1, 2, ..., n_x$, $l = 1, 2, ..., n_y$ and $a = -b = -\pi/2$ in the case of the weakly deformed domain Ω_1 and $a = -b = -1.9\pi/2$ in the case of the strongly deformed domain Ω_3. The function in (9) is used in order to allow the computations of the scalar products $(\sigma(x, y)\psi_{k,l}(x, y),\ \psi_{m,n}(x, y))$ as the Cartesian product of one dimensional integrals. These integrals have been computed separately for the variable x and variable y by using the Gauss quadrature formula of order 20 used on each subinterval $\langle x_i, x_{i+1}\rangle$, where x_i are the B-spline knots in the interval $\langle a, b\rangle$ in the case of the variable x. The resulting matrix eigenvalue problems of order $n(n = n_x \cdot n_y, n_x = n_y)$ have been solved by the subroutine NGHOUD from the FORTRAN package NICER [1]. The convergence of the computed eigenvalue approximations to exact eigenvalues of the equation (3) is prooved in [4].

The Rayleigh-Ritz eigenvalue approximations of the selected eigenvalues using n trial functions are presented in Table 1 – 4. For the sake of convergence comparisons the eigenvalue approximations shown in Table 1 and Table 2 are taken from the author's previous article [5] and correspond to the sine trial functions. The eigenvalue approximations shown in Table 3 and Table 4 are computed using the B-spline trial functions (9). The results reported in Table 1 and Table 3 correspond to the case of weakly deformed domain Ω_1, while the results reported in Table 2 and Table 4 correspond to the case of strongly deformed domain Ω_3.

Eigenvalue approximations for the weakly deformed domain Ω_1 using n = 400, 900, 1600 and 2500 sine trial functions. Tab. 1

n	λ_1	λ_2	λ_3	λ_5	λ_7	λ_{10}
400	7.5705280	15.221970	22.197600	29.167270	46.127290	66.425740
900	7.5698450	15.220040	22.196280	29.164740	46.120670	66.413520
1600	7.5696870	15.219590	22.195970	29.164170	46.119410	66.411250
2500	7.5696330	15.219440	22.195870	29.163970	46.119000	66.410529

Eigenvalue approximations for the strongly deformed domain Ω_3 using n = 400, 900, 1600 and 2500 sine trial functions. Tab. 2

n	λ_1	λ_2	λ_3	λ_5	λ_7	λ_{10}
400	53.829210	56.155410	141.61980	206.07930	311.82790	370.58720
900	37.753730	38.556880	98.792580	132.33440	207.75130	236.73640
1600	31.175410	31.522150	80.903980	101.72670	164.54580	182.51010
2500	27.901250	28.062540	71.828140	85.913150	142.56580	154.51040

Eigenvalue approximations for the weakly deformed domain Ω_1 using n = 400, 900, 1600 and 2500 B-spline trial functions (9) of order 8. Tab. 3

n	λ_1	λ_2	λ_3	λ_5	λ_7	λ_{10}
400	7.5695770	15.21928	22.195761	29.163773	46.118610	66.409851
900	7.5695770	15.21928	22.195761	29.163773	46.118610	66.409843
1600	7.5695770	15.21928	22.195761	29.163773	46.118610	66.409843
2500	7.5695770	15.21928	22.195761	29.163773	46.118610	66.409843

Eigenvalue approximations for the strongly deformed domain Ω_3 using n = 400, 900, 1600 and 2500 B-spline trial functions (9) of order 8. Tab. 4

n	λ_1	λ_2	λ_3	λ_5	λ_7	λ_{10}
400	26.851726	26.925258	78.485629	81.974518	164.708914	192.756097
900	23.345010	23.392738	60.415130	63.837916	124.972830	127.908500
1600	23.029351	23.074449	58.217327	59.586448	108.096674	115.765853
2500	23.007218	23.052105	58.047942	58.841866	104.813595	107.866751

5. Concluding remarks

The presented numerical results indicate that the *B*-spline trial functions offer more precise eigenvalue approximations than the sine trial functions. This difference in the eigenvalue convergence is caused by the presence of shape singularities of the exact eigenfunctions which are the consequence of very steep gradient of the function $\sigma(x, y)$ in the vicinity of the points $[-\pi, 0][\pi, 0]$ as seen in Figure 2 (on the right). This case corresponds to the strongly deformed domain Ω_3. Because the *B*-spline trial functions are able to match singular behaviour of functions more precisely than approximations based on the sine trial functions, the corresponding eigenvalue approximations exhibit essentially better convergence. Finally the *B*-spline trial functions are recommended for use at least in the cases when the domain with complicated boundary shape is generated by a conformal mapping of square or rectangle.

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