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LOCATION PROBLEMS IN TRANSPORTATION NETWORKS

It is known that many optimisation problems on networks are NP-hard. However, it seems that the real transport networks have some interesting properties which allow us to find a „good" solution in reasonable time. In this paper, we suggest and study some new parameters of the transportation networks which could be useful in optimisation problems. We define the evenness and the robustness of the solution. We also concern ourselves with the statistical distribution of distances and edge values in transportation networks.

Keywords: *Transportation networks, Euclidean networks, robustness, distribution of distances.*

1. Introduction and preliminaries

It is well known fact that many optimisation algorithms work more successfully in real networks than in random graphs. For example, the greedy method for the traveling salesman problem gives, in real transportation networks, better results than in general graphs [1]. The difference is caused by triangle inequality which usually holds in transportation networks. We can also mention networks with Euclidean metric. Their properties allow us to suggest more efficient algorithms [2].

We will need the following definitions from [3 and 4]. Network is an ordered quadruple $G=(V,E,c,w)$, where $V \neq \emptyset$ is the set of vertices, $E \neq \emptyset$ is the set of edges (oriented or un-oriented), $c: E \to R_0^+$ is a function which represents the length of edges and $w: V \to N_0$ represents the weights of vertices. Let $d_G(u,v)$ be the distance between vertices *u* and *v* in network *G* and let $d_G(v,D)=\min\{d_G(v,u) \mid u \in D\}$ be the distance between a set $D \subset V$ and a vertex *v* in network *G*.

The eccentricity of a set *D* is
$ec_G(D)=\max\{d_G(v,D) \mid v \in V\}$.

The weighted eccentricity of *D* is
$ecc_G(D)=max\{w(v)\cdot d_G(v,D) \mid v \in V\}$.

The total distance of vertices of *G* from *D* is
$f_G(D)\sum_{v\in V} d_G(v,D)$

The total weighted distance of vertices of *G* from *D* is
$f_G^w(D)=\sum w(v)\cdot d_G(v,D)$.

Now we are able to define various location problems on network *G*.

Set $D \subset V$ is the *p*-center of *G*, if $|D|=p$ and $ec_G(D) \leq ec_G(D')$ for any *p*-element subset $D' \subset V$.

Set $D \subset V$ is the weighted *p*-center of *G*, if $|D|=p$ and $ecc_G(D) \leq ecc_G(D')$ for any *p*-element subset $D' \subset V$.

Set $D \subset V$ is the *p*-median of *G*, if $|D|=p$ and $f_G(D) \leq f_G(D')$ for any *p*-element subset $D' \subset V$.

Set $D \subset V$ is the weighted *p*-median of *G*, if $|D|=p$ and $f_G(D) \leq f_G(D')$ for any *p*-element subset $D' \subset V$.

Set $D \subset V$ is the anti-*p*-center of *G*, if $|D|=p$ and $ec_G(D) \leq ec_G(D')$ or any *p*-element subset $D' \subset V$.

Set $D \subset V$ is the *p*-maxian of *G*, if $|D|=p$ and $f_G(D) \leq f_G(D')$ for any p-element subset $D' \subset V$.

Definitions of weighted anti-*p*-center and weighted *p*-maxian are similar. It is known that the mentioned problems are *NP*-hard (if *p* is a part of the input) [5 and 6]. This is the reason why heuristic algorithms are usually used for finding a suboptimal solution [7, 8 and [9]. The only exception is the anti-*p*-center problem, which is polynomial – as it was shown in [10].

2. Robustness of a solution

In this section we introduce the robustness of a solution of various location problems. Let $D \subset V$ (where $|D|=p$) be a solution (not necessarily optimal) of a given location problem with objective function $\min h_G$ (where $h_G \in \{ec_G, ecc_G, f_G, f_G^w\}$). We can suppose that *k* edges of *G* are not rideable. We denote the set of these edges by *F*. It is reasonable to investigate the changes of objective function in network *G - K* , where *G - F*

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denotes the network obtained from G by deleting the edges from F. The k-robustness of set D for a given objective function h_G is

$$\min = \left\{ h_G(D)/h_{G-F}(D) \middle| F \subset E, |E| = k \right\}.$$

The problem of finding a k-robustness of D for function $ec_G(D)$ seems to be similar to the problem of edge k-connectivity in graphs which is solvable in polynomial time [11]. An exact polynomial algorithm or heuristic based on edge k-connectivity with tests on real data will be published in the next paper. The application of k-robustness is in facility location problems. For example, we need to locate p emergency medical stations. We have solutions $D_1, D_2, \ldots, D_a$ of similar quality. Then the robustness can be the next criteria for choosing one solution, because it says more about stability of the solution when there are parts of the network that are not rideable.

Similar concepts are studied for example in [12], where the authors deal with a facility (vertex) disruption, and [13], where the author studies the network reliability in general.

3. Evenness of a solution

In [2], the Euclidean metric, the triangle inequality and its impact on the solution of the p-median problem have been studied. It follows from the results that practical algorithms are more efficient in networks with these properties. We would like to continue in the research started in the above mentioned work. Hence, we suggest the study of three new parameters of solutions of various location problem (two of them are taken over from mathematical statistics). We would like to compare these parameters in networks and in graphs without the above mentioned properties. These parameters are the mean of vertex distances, variance and evenness. Let the set $D \subset V$ ($|D| = p$) be given. The arithmetic mean of distances (of vertices) in D is

$$\bar{d}_G(D) = \frac{2}{p(p-1)} \cdot \sum_{\substack{u \in D \\ v \in D}} d_G(u,v).$$

The variance of distances (of vertices) in D is

$$Var(D) = \frac{2}{p(p-1)} \cdot \sum_{\substack{u \in D \\ v \in D}} \left(d_G(u,v) - \bar{d}_G(D)\right)^2.$$

The evenness of D in G is

$$\varepsilon_G(D) = Var(D)/Var(V).$$

We suppose that it is possible to approximate the mean, variance and evenness of the optimal solution from the properties of networks with Euclidean metrics. If we have appropriate approximations of these values, then we can restrict the set of admissible solutions. Computation of the optimal solution could be faster than computation in general graphs. It is convenient to test the properties of optimal solutions in large networks for the anti p-center problem, since this problem is solvable in polynomial time [10].

4. Distribution of distances

In this section we develop some ideas from the work [2], where the authors solve the p-median model using standard linear programming (plus branch and bound) method and the vertex substitution heuristic. The types of networks studied in [2] include Euclidean networks (distances between points are measured by Euclidean metric), path networks (with triangle inequality) and random distance networks. It follows from the results that the above mentioned methods are more efficient in Euclidean networks and path networks (which are similar to the actual transportation networks) even if the methods were not suggested for these special types of networks. We suppose that the algorithms using specific properties of these networks could provide significantly better results.

Important properties could be the distribution of distances and the distribution of edge lengths. Our tests on the road network of Slovakia show the distribution of edge lengths which is in Fig. 1.

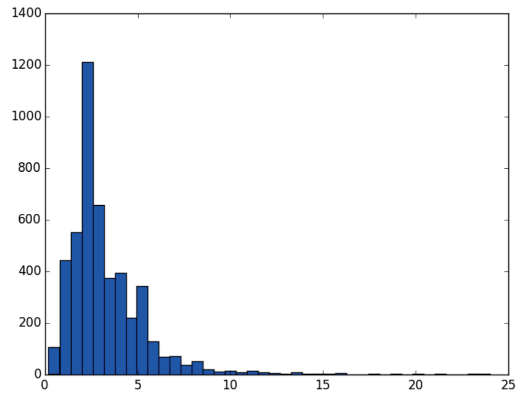

Fig. 1 The distribution of edge lengths in the road network of Slovakia

This distribution is similar to the Maxwell-Boltzmann distribution of molecule speeds in idealised gas [14] with the distribution function $f(x) = \sqrt{2/(\pi a^6)} \cdot x^2 \cdot e^{-x^2/(2a^2)}$, where a is an appropriate constant. The proof, that the distribution of edge lengths could be approximated by the Maxwell-Boltzmann distribution function, involves finding a special, continuous embedding of the road network into the three dimensional Euclidean space which preserves the incidence relation of a network and maps every edge to line segment with the same length. The existence of such embedding remains an open question. The next step of the proof involves finding the representation of the road network by the above mentioned physical model, where edge lengths mean speeds of the particles. We are preparing a paper which is devoted to this topic.

If we consider the distribution of distances in the road network, then we achieve a similar situation. The results for distance distribution in Euclidean networks can be found in [2] and, for distance distribution of bus stops and stations in China, can be found in [15], where the authors work with a gamma distribution. We think that the best approximation of the distance distribution can be found in [16], where the probability distribution for the distance between two random points in a rectangle with given sides is determined. Its probability density function for a rectangle with sides *a, b* is

$$f(x) = \frac{2x}{ab}\left(-\frac{2x}{a} - \frac{2x}{b} + \pi + \frac{x^2}{ab}\right), \text{ for } 0 < x \le a,$$

$$f(x) = \frac{2x}{b}\left(-\frac{2x}{a^2} - \frac{1}{b} + \frac{2}{a}\arcsin\left(\frac{a}{x}\right) + \frac{2}{a^2}\sqrt{x^2 - a^2}\right),$$

for $a < x \le b$,

and

$$f(x) = \frac{2x}{ab}\left(2\arcsin\left(\frac{a}{x}\right) + \frac{2}{a}\sqrt{x^2 - a^2} + 2arcin\left(\frac{b}{x}\right) + \right.$$
$$\left. + \frac{2}{b}\sqrt{x^2 - b^2}\right) - \frac{2x}{b^2} - \frac{2x}{a^2} - \frac{2\pi x}{ab} - \frac{2x^3}{a^2 b^2},$$

for $b < x \le \sqrt{a^2 + b^2}$.

We suppose that appropriate solutions of the *p*-center and *p*-median problem have the same distance distribution as the whole network. Our goal is the suggestion of a method for solving location problems which would be more effective on Euclidean and path networks than existing methods. Information about behaviour of solutions (distance distribution and evenness) can help us to find some new algorithms.

5. Conclusions

This paper is an introduction to our study of parameters and properties of transportation networks. We suppose that new knowledge in this area of research could bring more efficient methods for solving the problems that are too hard for exact computation in general.

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