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## ANALYSIS OF DYNAMIC PROPERTIES OF ACCELEROMETER USING FRACTIONAL DERIVATIVES

The paper outlines the use of fractional derivatives for dynamic measurements while developing a method of improved description of dynamic properties of accelerometer, which the authors consider to be the original and unique achievement of their work. The main objective of the work is an implementation of a fractional calculus-based method which allows for a description of dynamic properties of signal processing of measuring transducers with fractional orders. A dynamics of transducer is identified by the ARX method (AutoRegressive with EXternal input identification method). Identification of accelerometer was accomplished with the use of the MATLAB&Simulink package.

**Keywords:** Accelerometer, ARX method, fractional calculus, fractional derivatives, MATLAB&Simulink, measuring transducer.

### 1. Introduction

The problem of describing dynamic properties of objects by means of fractional calculus although well known since the times of Gottfried Wilhelm Leibniz [1], and [2], yet due to restrictions resulting from lack of appropriate calculation methods and possibilities of their verification has always been ignored.

At present technical and calculation possibilities cause that the problems related to these limitations have, to a large extent, been solved. There are more and more publications dealing with the topic of fractional order differential equations.

Majority of them, however, deal with theoretical aspects of the problem. There are no publications which put strong emphasis on the practical application of fractional calculus and combine theory with real applications. The paper outlines the use of fractional calculus for dynamic measurements while developing a method of improved description of dynamic properties of measuring transducers which the authors considers to be the original and unique achievement of this work. Proposed by the authors of this paper method of description of the dynamic properties of measuring transducer in terms of signal processing, based on fractional calculus, allows for a description of dynamic properties of broader class of measuring transducers, i.e. integer-order and fractional-order accelerometers. The aim of this paper is to investigate how models of accelerometers based on the fractional calculus ([3], [4], [5], [6], [1], [7] and [2]) description convey their dynamic behaviour in comparison to models represented by differential equations of integer orders

and in comparison to processing characteristics of their real counterparts.

### 2. Model of the second order accelerometer

Simulation and laboratory testing of the second order accelerometer is shown in Fig. 1.

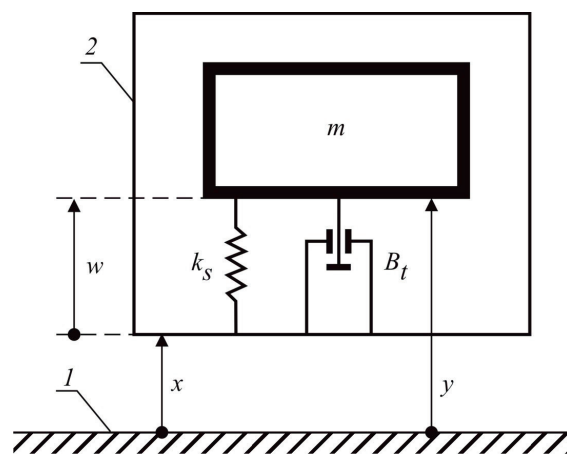


Fig. 1 Kinetic diagram of an accelerometer:  $m$  - vibrating mass,  $k_s$  - spring constant,  $B_t$  - damping coefficient,  $x$  - object motion relative to a fixed system of coordinates,  $y$  - motion of a vibrating mass relative to a fixed system of coordinates,  $w$  - motion of a vibrating mass relative to a vibrating object [4] and [7]

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A differential equation describing the absolute motion of the second order measuring vibrating mass of transducer can be expressed as [7]:

$$\frac{d^2}{dt^2}y(t) + 2\zeta\omega_0 \frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 x(t) + 2\zeta\omega_0 \frac{d}{dt}x(t) \quad (1)$$

where:

$$\omega_0 = \sqrt{\frac{k_s}{m}} \quad (2a)$$

$$\zeta = \frac{B_l}{2\sqrt{k_s m}} \quad (2b)$$

Considering the motion of the vibrating mass relative to the vibrating object (Fig. 1):

$$\frac{d^2}{dt^2}w(t) + 2\zeta\omega_0 \frac{d}{dt}w(t) + \omega_0^2 w(t) = -\frac{d^2}{dt^2}x(t) \quad (3)$$

Depending on the selection of  $k_s$ ,  $m$  and  $B_l$ , a transducer can serve to measure acceleration as an accelerometer [7] assuming a high  $k_s$ , a low  $m$  and  $B_l$ . In practical vibration measurements, acceleration-measuring transducers, the so-called accelerometers, are employed. For purposes of simulation testing, a measuring transducer was assumed of the frequency  $f = 350$  Hz, that is, circular frequency of free vibrations  $\omega_0 = 2200 \frac{rad}{s}$  and degree of damping  $\zeta = 0.2$ . Dynamics of such a transducer, characterized by means of the second order differential equation (3), is described by operator transmittance:

$$G(s) = \frac{1}{s^2 + 880s + 4.84 \cdot 10^6} \quad (4)$$

Figure 2 illustrates amplitude and phase frequency characteristics of a measuring transducer with operator transmittance (4).

### 3. Quasi-fractional model of accelerometer

There are many definitions of fractional derivative. Three most commonly used definitions are those by Riemann-Liouville, Grünwald-Letnikov and Caputo ([8], [1], [7] and [2]). The Grünwald-Letnikov derivative is:

$${}_t D_t^{(v)} f(t) = \lim_{h \rightarrow 0} \frac{1}{h^v} \sum_{i=0}^k a_i^{(v)} f(t - hi) \quad (5)$$

with:

$$a_i^{(v)} = \begin{cases} 1 & i = 0 \\ (-1)^i \frac{v(v-1)(v-2)\dots(v-i+1)}{i!} & i = 1, 2, 3, \dots \end{cases} \quad (6)$$

where (6) is defined as the reverse difference of the discrete function and  $h$  is the increment of  $f(t)$  defined in the range  $[t_0, t]$ :

$$h = \frac{t - t_0}{k} \quad (7)$$

Introducing a non-integral order to the measuring transducer's equation (1) converts it into:

$$\frac{d^2}{dt^2}w(t) + 2\zeta\omega_0 \frac{d}{dt^{(v)}}w(t) + \omega_0^2 w(t) = -\frac{d}{dt}x(t) \quad (8)$$

where  $v$  is order of a fractional derivative.

The concept of this work is based on a comparison of different models of an accelerometer's dynamic behaviour (based on differential equations of integer and fractional orders) with the processing characteristics of a real accelerometer so as to obtain answer to the question about which method of modelling is more accurate and whether there are any criteria for which a certain model is better at reproducing the dynamic behaviour of the real accelerometer. The research results has included algorithms of determining models describing an accelerometer's dynamic behaviour based on differential equations of integer and fractional orders for the definitions by Grünwald-Letnikov (5). The results were presented in works [2], [3], [4], [7] and [8].

On comparing responses of the measuring transducer to the input sinusoid signal, it was described by means of three models using the Grünwald-Letnikov derivative (5): *Classic model* described with operator transmittance (9) - the same as formula (4). The operation of a transducer described with the equation was simulated for appropriately selected parameters: natural angular frequency and degree of damping:

$$G(s) = \frac{1}{s^2 + 880s + 4.84 \cdot 10^6} \quad (9)$$

*Classic discrete mode*, derived from the operator transmittance model (9), described by means of discrete transmittance (10):

$$G(z) = \frac{1.667 \cdot 10^{-15} z^2 + 6.666 \cdot 10^{-15} z + 1.667 \cdot 10^{-15}}{z^2 - 2z + 0.999} \quad (10)$$

Response of a continuous object to a discrete input depends not only on values of this signal at discrete moments of time but also on sampling time and the extrapolator used.

The *quasi-fractional discrete model* described by discrete transmittance (11):

$$G(z) = \frac{z^2}{1 \cdot 10^{14} z^2 - 2 \cdot 10^{14} z + 1 \cdot 10^{14}} \quad (11)$$

Discrete transmittance (11) is produced by implementation of the method of determining quasi-fractional expression of the measuring transducer in MATLAB&Simulink programme [3], [5], [7] and [9]. It can be noted that the model described by means of the discrete transmittance (11) correctly reproduces values of the input signal amplitude - the same as the model of

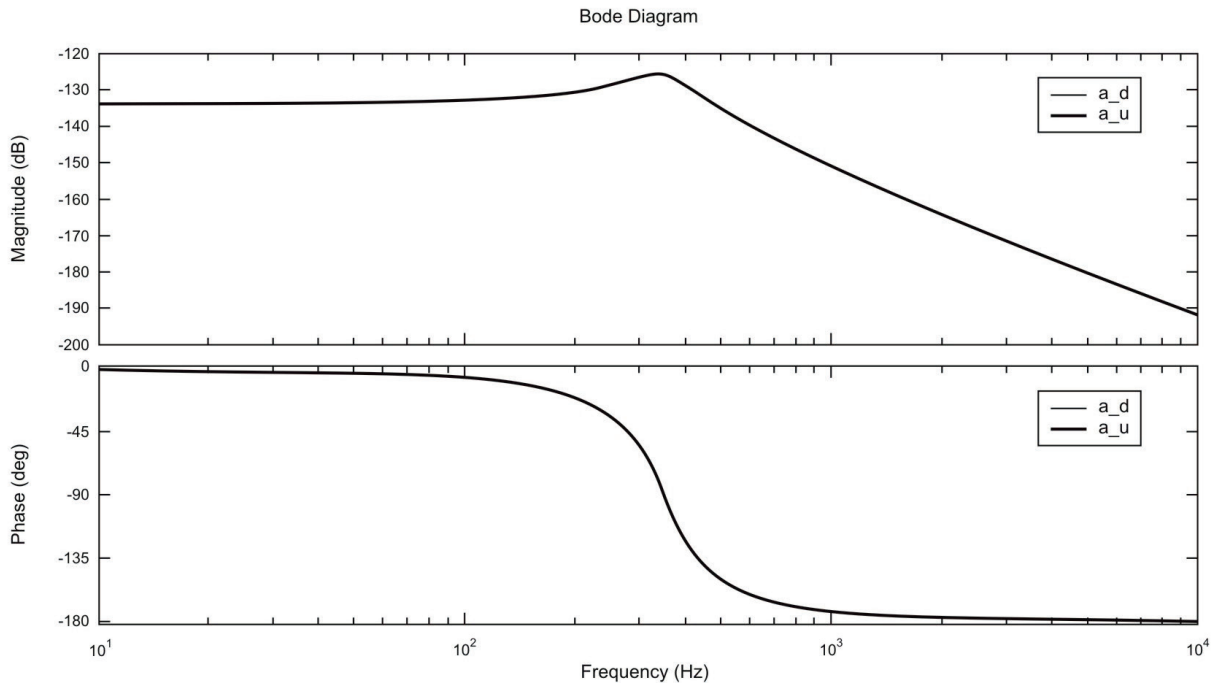


Fig. 2 Comparison of Bode diagrams of measuring transducer models (10) and (11) (diagrams of the models overlap) [4]

transmittance (10). It can be noted in Bode frequency diagrams [6] (Fig. 2) that the measuring transducer model determined by the derivative-integral method presents the dynamics of the classically determined model (the diagrams of the models coincide). This confirms that integral-order differential-integral calculus is a special case of differential-integral calculus of non-integral orders.

Measuring transducer models (10) and (11) have only been subject to simulation testing and do not fully represent real models but the simulations indicate that the quasi-fractional model (11) exhibits the same dynamics as the classic model. The ‘apparent’ time of stabilization of time diagrams – the time after which description of a model is independent from time – for the quasi-fractional model is the same as for the classic model.

#### 4. Model of a laboratory system with accelerometer

An overview of the measurement system is shown in Fig. 3. In order to determine measuring transducer’s operator transmittance, we built a system with two accelerometers *A1* and *A2*. Accelerometer *A1*- DeltaTron by Bruel&Kjaer type 4507, which sensitivity 10.18mV/ms<sup>2</sup> and the range of frequency measurements from 0.4 Hz to 6 kHz was tested. The operating ranges of conditioner was between 1Hz and 20 kHz. The transducer was mounted on an electrodynamic inductor. A model accelerometer *A2* - by VEB Metra, type KB12, which sensitivity 317 mV/ms<sup>2</sup> was aligned with the tested transducer *A1*.

The operator transmittance (12) describing dynamics of the measurement system was determined by identification with an

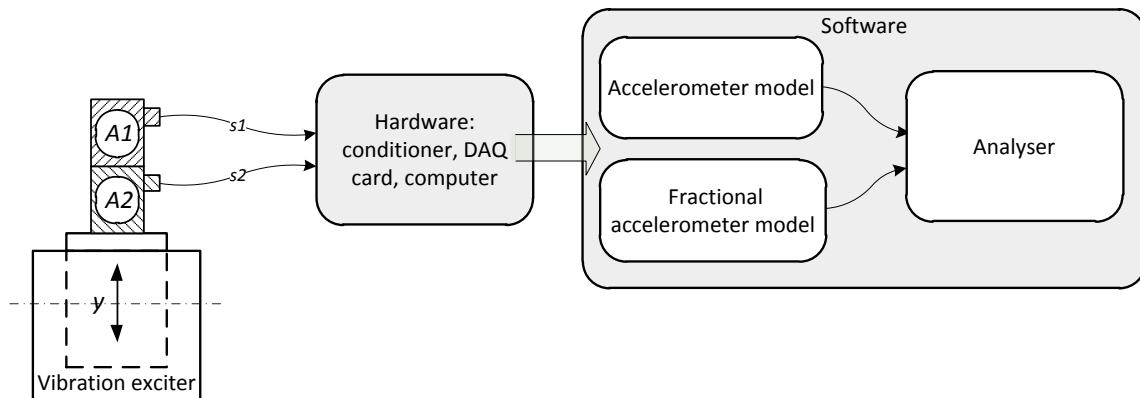


Fig. 3 Laboratory measurement system for testing of mechanical vibration transducers: *A1*, *A2* - model and tested measuring transducers [9]

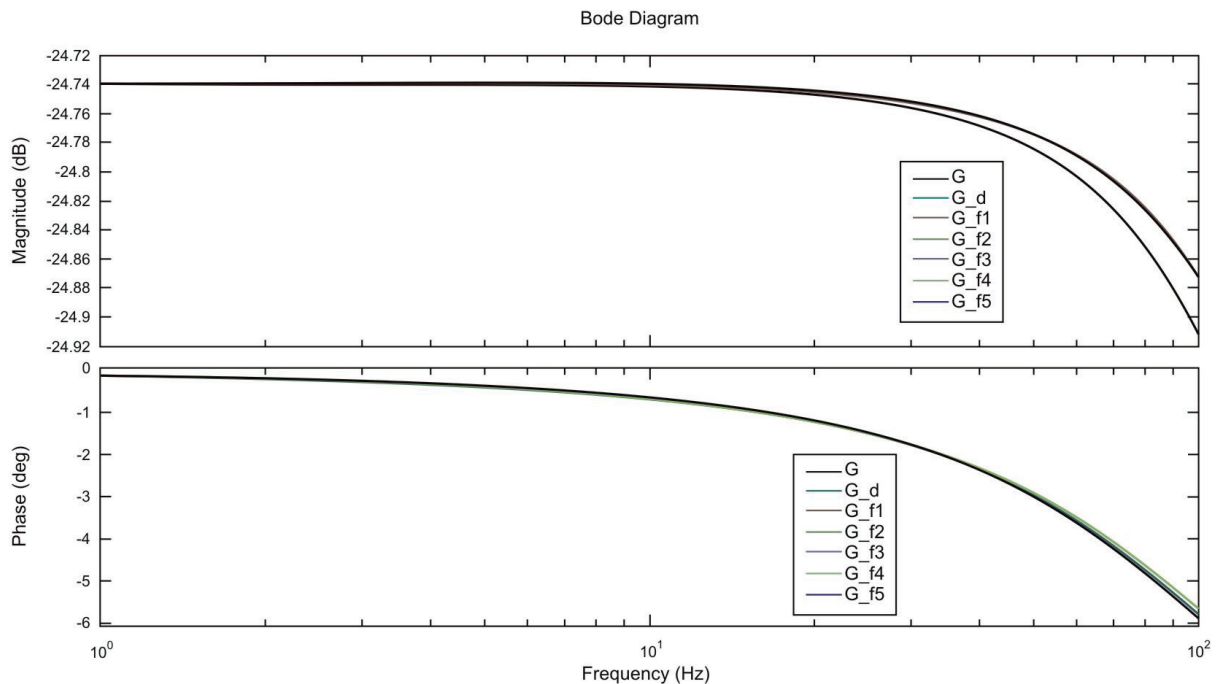


Fig. 4 Bode frequency diagrams of measuring transducer models of transmittance (models from the Table 1) [4]

external ARX (AutoRegressive with EXternal input) [7]. The voltage signal from the end of the tested measurement track is the identified signal, and the signal from the model accelerometer in response to the sinusoidal function of the generator of 100 Hz is the comparative signal.

The ARX identification method produced the operator transmittance  $G(s)$  describing the dynamics of the system:

1. Classic model:

$$G(s) = \frac{0.03215s^2 + 1319.6s + 1.338 \cdot 10^6}{s^2 + 4.678 \cdot 10^4 s + 2.309 \cdot 10^7}. \quad (12)$$

2. Discrete transfer function of the model was determined on the basis of the operator transmittance (12):

$$G(z) = \frac{0.03215z^2 + 0.05368z + 0,02163}{z^2 + 1.625z + 0.6264}. \quad (13)$$

The discrete model (13) was produced by discretizing the classic model (20) by means of the 'Zero-Order-Hold' method [7] with the sampling time  $T_p = 10^{-4}s$ .

3. Discrete transfer function of fractional models was determined with a method implemented in MATLAB&Simulink. For varying increment of  $h$ , quasi-fractional transducer models become discrete transmittances which is presented in the Table 1 [4]:

Discrete transmittances of measuring transducer models for varying increment Table 1

Increment variation $h$	Discrete transmittance $G(z)$
$10^{-7}$	$G_{f_1}(z) = \frac{3.228z^2 - 6.443z + 3.215}{100.5z^2 - 200.5z + 100}$
$10^{-6}$	$G_{f_2}(z) = \frac{3.347z^2 - 6.562z + 3.215}{104.7z^2 - 204.7z + 100}$
$10^{-5}$	$G_{f_3}(z) = \frac{4.548z^2 - 7.75z + 3.215}{104.7z^2 - 246.8z + 100}$
$10^{-4}$	$G_{f_4}(z) = \frac{1.775z^2 - 1.963z + 0.322}{59.09z^2 - 66.78z + 10}$
$10^{-3}$	$G_{f_5}(z) = \frac{2.69z^2 - 1.384z + 0.032}{70.87z^2 - 48.78z + 1}$

The model of the real measurement system in the form of discrete transmittance and models expressed by means of a differential-integral equation were then compared. Both types of the models were based on the classic model derived by ARX identification method.

Figure 4 shows logarithm frequency amplitude and phase diagrams of the measurement system models. It can be observed that for the adopted increment of  $h$ , the diagrams clearly diverge. This means that other increments  $h$  far lower than the sampling frequency, must be adopted.

## 5. Conclusion

Dynamic development of recent research into the use of fractional calculus for the dynamic system analysis encouraged the authors of this paper to attempt the use fractional derivatives for the analysis and modelling of measuring transducers and measurement systems [3], [4], [5], [7] and [9]. The main objective of the work is an implementation of a fractional calculus-based method for a description of dynamic properties of signal processing of measuring transducers with integer-order and quasi-fractional-order. Fractional calculus is a generalisation of integral-order differential calculus – this is confirmed by laboratory testing of dynamic systems.

The effect of  $h$  on fractional measuring amplitude of fractional measuring transducer and phase diagrams was also shown in this paper. In this case increments  $h$ , far lower than the sampling frequency, must be adopted [4].

Application of the quasi-fractional method of describing dynamic properties of measuring transducers discussed in this paper, based on fractional calculus, will help to undertake analyses of simulated dynamics of various objects and processes which, due to their complexity, must be described by means of differential equations of any orders.

## References

- [1] OSTALCZYK, P.: *Epitome of the Fractional Calculus. Theory and its Applications in Automatics (in Polish)*, Wydawnictwo Politechniki Lodzkiej, 430 p., ISBN 978-83-7283-245-0, Lodz, 2008.
- [2] PODLUBNY, I.: *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications*. Academic Press, 368 p., ISBN 0125588402, San Diego: Boston: New York: London: Tokyo: Toronto, 1999.
- [3] LUFT, M., CIOC, R., PIETRUSZCZAK, D.: *Fractional Calculus in Modelling of Measuring Transducers. Electronics and Electrical Engineering*, No. 4(110), ISSN 1392-1215, Kaunas, 2011.
- [4] LUFT, M., SZYCHTA, E., PIETRUSZCZAK, D.: *Some Applications of Fractional Calculus to the Analysis of Dynamic Properties of Selected Measuring Transducer*. Proc. of 11<sup>th</sup> European conference of young scientists and postgraduate students Transcom 2015, section 4, Electric Power System, Electrical and Electronic Engineering, ISBN 978-80-554-1046-3, pp. 29-34, June 2015, Zilina, 2015.
- [5] LUFT, M., SZYCHTA, E., CIOC, R., PIETRUSZCZAK, D.: *Application of Fractional Calculus in Identification of the Measuring System*. Transport Systems and Processes, CRC Press Balkema, Taylor & Francis Group, pp. 63-68, ISBN 978-0-415-69120-8, London, 2011.
- [6] LUFT, M., SZYCHTA, L., NOWOCIEN, A., CIOC, R., PIETRUSZCZAK, D.: *Bode Plots of the Pressure Sensor Modelled by Fractional Calculus (in Polish)*, *Logistyka*, 4/2015, ISSN 1231-5478, pp. 4539-4545, Poznan, 2015.
- [7] PIETRUSZCZAK, D.: *Application of Fractional Calculus to the Analysis of Dynamic Properties of the Measurement Systems (in Polish)*, Doctoral dissertation, The Main Library of Kazimierz Pulaski University of Technology and Humanities, Radom, 2012.
- [8] KACZOREK, T.: *Selected Problems of Fractional Systems Theory*. Springer-Verlag GmbH, 344 p., ISBN 978-3-642-20501-9, Berlin, 2011.
- [9] PIETRUSZCZAK, D., SZYCHTA, E.: *Analysis of Selected Dynamic Properties of Fractional Order Accelerometers for Application in Telematics Equipment*. Communications in Computer and Information Science, ISSN 1865-0929, pp. 321-329 Springer-Verlag Berlin Heidelberg, 2013.