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ECONOMICALLY OPTIMAL ROAD SUBNETWORK

The paper deals with the following situation: An area is served by a transportation network, usually the road one. The current quality of the network is not satisfactory for the owner (e.g. a company or public administration), but the reconstruction or recovery of the whole network is not feasible for the economic reasons. Managers, who are responsible for the network, make decisions how to reduce the network and then to reconstruct or recover it meeting certain conditions and minimizing costs. The condition is formulated by means of the set W of important pairs of sources and sinks of transport flows and by the number $q \geq 1$ representing the maximal acceptable elongation rate of routes between nodes (vertices) from the set W .

Such a problem can be met e.g. in rural road network reduction for winter maintenance, choice of tram or trolleybus network as a sub-network of the bus one etc.

The paper describes the mathematical support for that decision making. The mathematical model of the problem is presented. Then a depth-first search type exact method is proposed and verified. Afterwards, a heuristics is described and verified as well. Finally, linear programming version of the problem is added.

The results were applied to urban bus network of Slovak town Piestany.

Keywords: Road, network, subnetwork, optimization.

1. Introduction

The paper describes the results of research focused on mathematical support for the following managerial decision-making problem: For the given road network, select a subnetwork that would be enough to fulfill a mission and whose adaptation or reconstruction would require minimal resources. One can mention several examples.

1.1 Rural Roads Network

As can be seen from the road map, in the Czech Republic (and certainly somewhere else), one can find many regions where the average distance between neighboring villages is about 2km and there are direct roads from each village to other 3-6 villages – see e.g. areas around Cheb, Kolín, Telc etc. Such a network is too long for the regular maintenance, mainly in the winter time. The problem is to reduce it and to determine a suitable subnetwork. It is desirable not to increase the length of routes for strong vehicle flows at all, or to do it only partially.

1.2 Urban Trolleybus or Tram Network

The usual development of public transport in medium towns begins by the phase of bus transport and then, after many years, the idea of partial electrification arises. The problem is how to choose a trolleybus or tram subnetwork of the bus network. Naturally, the electrified subnetwork has to serve to the strongest passenger flows and not to force them to travel on much longer routes than before. The partial change of bus to trolleybus networks took place e.g. in Slovak towns Zilina (1994), Banská Bystrica (1989), in Czech towns Chomutov (1995), Opava (1952).

1.3 Urban Bus Transport Intensification

In the past, in several Czech and Slovak towns with 30-60 thousands inhabitants, the bus urban transport was of ‘extensive’ type. I.e., on almost every street ran a bus line, but the intervals between successive buses were very long, ½-1 hour or even more. Further, the city administration ought to build and maintain many stop shelters, adapted for long waiting passengers. Some town hall managers were displeased

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with it. They decided to intensify the system by reducing the network, but with the condition that the travel time extends for more than 15% for no significant passenger flow. Such an arrangement saves time for passengers and money for the town hall, since the system needs less stops with smaller and cheaper stop shelters - see [1].

1.4 Other Transport Networks

Other transport networks, e.g. rural railways, continental waterways, mountain ski tows and cableways, may be subject of reduction because of small demand or necessity of expensive modernization. However, it is desirable not to increase the length of routes for strong passenger (or freight) flows at all, or to do it only partially.

1.5 Common Features of the Decision Problems

All four abovementioned decision problems have several common features:

- A network, i.e. a non-oriented graph, is given.
- Each edge of the network has a 'length' in a general sense - for example, it may be the actual length, the time of transit, the cost of construction or maintenance, etc. The notion of 'length' is extended to the set of routes as the sum of lengths of the edges forming the route.
- A 'demand flow' is given for each pair of vertices. Several of the pairs are considered important, e.g. the ones with high flow between them.

The decision problem is to find a subnetwork of minimal 'length' where no important pair of vertices has a distance (= the 'length' of the shortest path) exceeding $q \cdot d$, where $q \geq 1$ is the given number and d is the distance of the pair on the original network.

1.6 Optimal Subgraph Problem in Bibliography

The decision problem formulated in 1.5 belongs to the family of network design problems (NDP's). One can look at it from several perspectives. The civil engineering will ask how to construct them. The geographical one focuses attention on the accessibility of network, i.e. the distance of the origin of the trip to the closest entrance and the distance of the closest exit to the destination. The transport viewpoint observes transport time, reliability, costs etc. The mathematical questions are what model to create or choose and what method to use.

The "classic" NDP [2] can be formulated as follows: Given a transport network and a transport demand on it, the problem is to improve current links (= edges) or to add new ones minimizing

the total (and usually generalized) travel costs of all demand elements. A NDP is said discrete [3] if it is focused on the link addition to an existing transport network. A NDP is said continuous [4 and 5] if it touches the improvement of current links. A particular case of public transport system is dealt in [3], [6] present the bimodal bus&car discrete urban road network design problem of bi-objective type: lane addition to the existing streets, new street constructions, converting some two-way streets to one-way streets, lane allocation for two-way streets, and the allocation of some street lanes for exclusive bus lanes are the studied measures focused on maximization of both consumer surplus, and maximization of the demand share of the bus mode.

Geographically, a typical feature of modern transport systems is the growth of networks [7]. However, it brings not only many positive effects, but also some negative ones. For instance, [8] draw attention to the rise in trip length and travel time in the metropolitan area of Madrid. On the other hand, [9] show that there are measures against this negative development.

Until now, there was presented the "main stream" in the theory of NDP dealing with the growth of networks, improving current links or adding new ones. However, that doesn't apply in some weak demand areas, e.g. in remote regions. There, one can see the opposite phenomenon - the need to reduce the existing network, e.g. for economic reasons or for the reasons mentioned in 1.1 - 1.4. And that is the main topic which is dealt below using the abbreviation NDP-R (Network Design Problem - Reduction). Of course this may increase the risk of network vulnerabilities, as pointed out by [10].

To find an article oriented directly to the NDP-R is difficult, they are rare. One can mention the paper by [11]. It starts with possible, i.e. not yet existing, network. The problem is to select a subnetwork to be really constructed minimizing the total construction and transportation costs.

It is obvious that, in general, the NDP-R can be solved by the models and techniques described in the papers mentioned above, provided that the constraints and objective function remain unchanged. However, the following problem does not fall into this category.

Suppose an area is covered by a transportation network. This coverage is thought ideal in terms of meeting transport demand. However, it is too expensive to operate the entire network. Therefore, it is necessary to reduce it to the cheapest subnetwork "sufficient to fulfill the original purpose". The last condition is formulated by means of the set W of important pairs of sources and sinks of transport flows and by the number $q \geq 1$ (e.g. $q = 1.2$) representing the maximal acceptable lengthening rate of routes between nodes (vertices) from the set W .

In the sequel, there are presented graph-theoretical models and methods applicable to this particular problem of transport subnetwork optimization. Hence graphs are used in mathematical context and networks concern transport.

In the graph theory, a large family of problems related to construction of extremal (max or min) subgraphs with the given properties has been studied for decades. E.g., the first known algorithm designing the shortest spanning tree was published in 1926 [12]. The further development of spanning tree theory is described in [13].

Typical representatives of the family are the following problems starting with the given graph $G = (V, E, c)$ where $c(e) \geq 0$ is a “cost” or “weight” (e.g. length or passing time or toll) of the edge $e \in E$:

- find the maximum planar subgraph [14],
- given integer $\lambda > 0$, find a λ -edge connected spanning subgraph with the minimum sum of edge weights [15 and 16] and also [17] where $c(e) \equiv 1$ and [18] where $\lambda = 2$,
- given integer $k > 0$, find a k -vertex subgraph with the maximum sum of edge weights [19],
- given integers $k > 0, \lambda > 0$, find a k -vertex λ -edge connected subgraph with the minimum sum of edge weights [20 and 21],
- given integer $d > 0$, find a subgraph with vertex degree not exceeding d with the maximum sum of edge weights [22],
- given integer $d > 0$, find a subgraph with vertex degree not less than d with the minimum sum of edge weights [22],
- given a subgraph H of G , find the minimum weight-sum subgraph isomorphic to H [23],

Other family members differ a bit from the ones mentioned above. E.g., [23] have weighted vertices instead of edges. Their second problem is to find the minimum weight-sum subgraph isomorphic to the given graph H . Or [24] considers $c(e)$ as the probability of defect (= break of service) and the problem is to eliminate k edges in order to maximize the connection reliability. On the other hand, [25] study a finite set of graphs G_1, \dots, G_n and their problem is to find the maximum common subgraph, while [26] seek the maximum acyclic subgraph of a digraph.

2. Optimization Problem

All the examples 1.1 - 1.4 may lead to the following mathematical **problems**:

2.1 Subnetwork for Complete Flow Matrix

In this case one can suppose that for each pair of nodes (v, w) a flow from v to w may exist.

Let $G = (V, E, d)$ be a connected undirected finite graph without loops with the length $d(e)$ for each $e \in E$. Let $d(p) = d(v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n) = d(e_1) + \dots + d(e_n)$ be the length of the path p and let $d(v, w)$ be the distance between the vertices v and w on the graph G , i.e., the length of the shortest path from v to w .

Let $q \geq 1$ be a real number called the upper bound of the length extension ratio.

The problem is to find a graph $G' = (V, E', d')$ such that

C1: $E' \subset E, d'(e) = d(e)$ for each $e \in E', d'(v, w)$ is the vertex distance on the graph G' .

C2: $d'(v, w) \leq qd(v, w)$ for each pair $(v, w) \in V \times V$

C3: $\sum_{e \in E'} d(e) \rightarrow \min$

2.2 Subnetwork for Important Flows

In this case one can suppose that there exists a set W of selected pairs of vertices (v, w) with important flow from v to w .

Let $G = (V, E, d), d(e), d(p), d(v, w)$ and q have the same meaning as in 2.1

Let $\emptyset \neq W \subset V \times V$ where each $(v, u) \in W$ represents “important flow direction” from the vertex v to the vertex u (the other flows are so small that they are neglected in the problem)

The problem is to find a graph $G' = (V, E', d')$ such that

C1: $E' \subset E, d'(e) = d(e)$ for each $e \in E', d'(v, w)$ is the vertex distance on the graph G' .

C2: $d'(v, w) \leq qd(v, w)$ for each pair $(v, w) \in W$

C3: $d(E') = \sum_{e \in E'} d(e) \rightarrow \min$

It is easy to see that 2.1 is a particular case of 2.2, one only needs to replace the set W by the set V . Any method solving 2.1 is suitable for 2.2 as well.

2.3 Other Similar Problems

It is possible to modify the basic problem 2.1 in a way different from 2.2.

- One can change the role of the constraint on the length extension ratio and the objective of the total length of the subnetwork and look for a subnetwork with a limited length minimizing the maximum of the length extension ratio.
- One can replace the previous min-max objective by the weighted sum where the coefficients are proportional to the flows.

3. Exact Methods of Solution

It is known [3] that the discrete network design problem has been recognized as one of the most difficult yet challenging problems in transport. As shown in [27], both problems 2.1 and 2.2 are NP-hard. Therefore, even in the case of Problems 2.1 and 2.2, the exact solution can be expected only for small instances of them - say, until 25 vertices.

In the chapters 3 and 5, two basically different exact methods are described. In the chapter 6, it is shown that these methods are usable for different networks with 20 vertices and sometimes even

with 30 ones. In the chapter 4 an original heuristics is presented. All these methods are compared in the Table 1.

At the beginning, it is good to predict whether the resulting set E' will have the number of elements $|E'|$ closer to $|E|$ or to 0. In the first case it is better to use the following decreasing method. In the second case the increasing one is better. If the prediction is not possible then one can start with any one of the methods (the authors prefer the decreasing one). Both these methods are of the "depth first search" type.

3.1 Decreasing Exact Method for the Problem 2.1

First, the ordering of the set $E = \{e_1, e_2, \dots, e_n\}$, $n = |E|$ is chosen. The computing experiments show that it can affect the duration of computations (see later).

Start with $E' = E$. Obviously the constraint C2 is valid and the record is $d(E') = d(E)$. Then subsequently inspect the sets $E' = E - \{e_1\}$, $E - \{e_1, e_2\}$, $E - \{e_1, e_2, e_3\}$ until the first moment when the constraint C2 fails, say, for $E' = E - \{e_1, e_2, \dots, e_{k-1}, e_k\}$. Then the set $E' = E - \{e_1, e_2, \dots, e_{k-1}\}$ is a temporary solution with the record $d(E')$. If $k = 1$ then the procedure stops since the optimal solution is obviously $E' = E$. If $k > 1$. Then the backtracking is done omitting e_k and replacing it by e_{k+1} etc.

3.2 Increasing Exact Method for the Problem 2.1

Start with $E' = \emptyset$. Obviously the constraint C2 is not valid. Then the sets $E' = \{e_1\}$, $\{e_1, e_2\}$, $\{e_1, e_2, e_3\}$ are subsequently inspected until the first moment when the constraint C2 is fulfilled, say, for $E' = \{e_1, e_2, \dots, e_k\}$ with the record $d(E')$. If $k = n$ then stop since the optimal solution is obviously $E' = E$. If $k < n$ then backtrack, omit e_k and replace it by e_{k+1} etc.

4. Heuristic Method of Solution

General NDP is solvable by several metaheuristics, e.g., ant colony system, tabu search, simulated annealing, genetic algorithm and their hybrids [28 and 29]. For the Problems 2.1 and 2.2, a new method is presented, based on the following lemma:

4.1 Lemma

Let $G = (V, E, d)$ be a connected undirected finite graph without loops with the length $d(e)$ for each $e \in E$. Let $d(p) = d(v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n) = d(e_1) + \dots + d(e_n)$ be the length of the path $p = v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n$ and let $d(v, w)$ denote the distance between the vertices v and w on the graph G . Let $E' \subset E$, let $q > 1$ and let

$$d'(e) = d(e)/q \text{ for each } e \in E' \text{ and } d'(e) = d(e) \text{ for } e \in E - E'$$

Let $v \in V$ and $w \in V$ be arbitrary. Let $p = v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n$ be the shortest path connecting the vertices v, w in the graph $G_0 = (V, E, d)$. Then $d(p) \leq q \cdot d(v, w)$.

Proof. Obviously

$$q \cdot d'(p) = q(d'(e_1) + d'(e_2) + \dots + d'(e_n)) \geq d(e_1) + d(e_2) + \dots + d(e_n) = d(p)$$

Let r be the shortest path connecting the vertices v, w in the graph G . Of course, in the graph G_0 it need not be true and thus $d'(p) \leq d'(r) \Rightarrow d(p) \leq q \cdot d'(p) \leq q \cdot d'(r) \leq q \cdot d(r) = q \cdot d(v, w)$ since $d'(e) \leq d(e)$ for each $e \in E$. The proof is complete.

4.2 The Heuristic Algorithm

- 1°: Put $j = 0$, $E_j = \emptyset$, $G_j = G$.
- 2°: Find the shortest path $p(v, w)$ on G_j for each pair $(v, w) \in V \times V$.
- 3°: For each $e \in E - E_j$ put $m_j(e) = \text{card}\{(v, w) \in V \times V : e \in p(v, w)\}$
- 4°: Find such $e^* \in E - E_j$ that $m_j(e^*) = \max\{m_j(e) : e \in E - E_j\}$.
 - a) If $m_j(e^*) = 0$ then put $E' = E_j$ and stop.
 - b) If $m_j(e^*) > 0$ then do 5°.
- 5°: Put $E_{j+1} = E_j \cup \{e^*\}$, $d_{j+1}(e^*) = d(e^*)/q$, $d_{j+1}(e) = d_j(e)$ for each $e \in E - \{e^*\}$, $G_{j+1} = (V, E, d_{j+1})$, add 1 to j ("put $j = j + 1$ ") and go to 2°.

Comments: The heuristic algorithm ends when 4°a) occurs. Then for each pair $(v, w) \in V \times V$ the shortest path $p = (v, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, w)$ connecting the vertices v and w in the graph G_j passes through $E' = E_j$ only (i.e. $e_1 \in E_j, \dots, e_n \in E_j$). Hence it is the shortest path between v and w in the graph $G' = (V, E', d)$, i.e. $d(p) = d'(v, w)$. On the other hand, the conditions of the lemma 4.1 hold, which implies $d'(v, w) \leq q \cdot d(v, w)$, i.e., the constraint C2 of the problem 2.1 is met and thus the set E' represents a feasible but not necessarily optimal solution of it.

If the heuristics 4.2 is modified in such a way that instead of "each pair $(v, w) \in V \times V$ " each pair $(v, w) \in W \times W$ is taken, the solution of the problem 2.2 is resulting.

5. Solution by Integer Linear Programming

In the next text, only the problem 2.2 is studied, since the problem 2.1 is its particular case.

Let $n \geq 2$ be a positive integer, let $V = \{1, 2, \dots, n\}$ and let $G = (V, E, d)$ be a connected undirected finite graph without loops with the length $d(e)$ for each $e \in E$. Further, let $d(v, w)$ be the distance between the vertices v and w on the graph G . Let $\emptyset \neq W \subset V \times V$ and finally let $q \in \langle 1, \infty \rangle$.

The problem is to find the values of the following integer variables:

$$x_{ij} \in \{0, 1\} \text{ for } (i, j) \in E, i < j, \tag{5.1}$$

$$y_{vw} \in \{0, 1\} \text{ for } (v, w) \in W, (i, j) \in E, \tag{5.2}$$

Table 1

TN	ONV	ONE	ONL	q	NIP	Exact method (EM)		Heuristic method (HM)		Integer linear programming (LP)	
						RNL	CT [s]	RNL	CT [s]	RNL	CT [s]
1	20	30	213.3	1.2	13	88.5	1596	91.2	< 1	88.5	3.5
				1.2	20	88.5	1053	88.5	< 1	88.5	11.4
				1.5	13	72.8	12430	82.5	1	72.8	2.4
				1.5	20	72.8	8844	79.8	1	72.8	8.6
2	20	30	204.2	1.2	13	65.8	5205	65.8	< 1	65.8	7
				1.2	20	65.8	3658	83.3	1	65.8	10.8
				1.5	13	54.9	7757	65.8	1	54.9	9.5
				1.5	20	65.8	1194	65.8	1	65.8	9.8
3	20	32	232.7	1.2	13	84.8	3033	89.5	1	84.8	7.9
				1.2	20	127.6	142	128.8	1	127.6	27.6
				1.5	13	78.6	146	89.5	1	78.6	8.9
				1.5	20	99.9	2681	122.4	1	99.9	31.8
4	20	30	204.6	1.2	13	74.1	2735	74.6	1	74.1	4.2
				1.2	20	103.0	226	105.3	1	103	17.5
				1.5	13	73.4	7884	74.6	1	73.4	5.1
				1.5	20	85.2	1906	87.4	1	85.2	19.7
5	20	29	209.6	1.2	13	87.7	485	99.7	< 1	87.7	7.8
				1.2	20	103.5	41	112.6	1	103.5	11.5
				1.5	13	80.3	2479	99.7	1	80.3	8.6
				1.5	20	90.4	555	120.2	2	80.3	19.2
6	20	32	212.9	1.2	13	92.5	3168	128	1	92.5	13.6
				1.2	20	96.6	1611	138.5	1	96.6	46.7
				1.5	13	81.4	13856	97.5	1	81.4	790.6
				1.5	20	96.6	7237	108.2	1	U	-
7	20	31	176.5	1.2	13	48.3	16672	54.1	1	48.3	10.5
				1.2	20	59.9	2491	65.7	1	59.9	41.5
				1.5	13	48.3	17912	54.1	1	48.3	8.1
				1.5	20	59.9	4818	65.7	1	59.9	44.8
8	20	28	197.7	1.2	13	81.8	900	95.7	1	81.8	4.3
				1.2	20	91.8	347	115.2	1	91.8	15.2
				1.5	13	66.3	1606	69.9	1	66.3	8.9
				1.5	20	66.3	730	66.3	< 1	66.3	17.9
9	20	30	191.6	1.2	13	115.2	11	137.9	2	115.2	6.8
				1.2	20	122.9	6	145.7	1	122.9	17.8
				1.5	13	75.3	2370	101.5	1	75.3	25.7
				1.5	20	93.3	240	125.5	1	93.3	64
10	30	47	271.9	1.2	13	U	-	95.8	1	78.1	15.9
				1.2	20	U	-	101.8	4	84	49
				1.5	13	U	-	70.1	1	58.8	32.7
				1.5	20	U	-	76.1	1	75.8	128.6

The legend: TN = Test network, ONV = Number of vertices, ONE Number of edges and ONL = Total length of the original network, NIP = Number of Important Pairs, RNL = Total length of the reduced network, CT = Computational time, U = Solution was not reached in an acceptable time

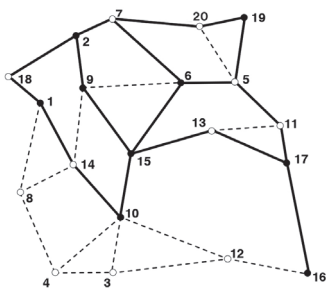


Fig. 4 Heur. solution - NIP = 20 and q = 1.2

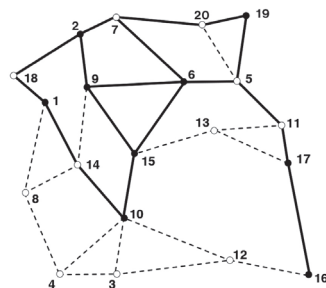


Fig. 5 Heur. solution - NIP = 20 and q = 1.5

Fig. 2 depicts the exact solution of the problem 2.2 in the case of $q = 1.2$ and Fig. 3 the same for $q = 1.5$. Analogically, Figs. 4 and 5 depict the same for the heuristics. It is remarkable, that the resulting subgraphs in Figs. 2 and 4 are almost equal. I.e., in this case, the heuristic solution for $q = 1.2$ is nearly the same as the optimal solution for $q = 1.2$.

7. Application in Practice

The abovementioned heuristic method was applied to the practical situation in the Slovak town of Piestany with about 30 thousands permanent residents and (at least) hundreds of spa guests. The number of vertices is 20, the number of edges is 31 and the number of important pairs of vertices (NIP) is 20. The network – see Fig. 6 – represents all streets that are suitable for bi-directional bus traffic on the right bank of the Vah River. It does not contain the connections to the western suburb from the vertex 14, to the northern suburb from the vertex 3 and to the spa area from the vertex 8, since these links must not be omitted by any reduction.

Figure 7 depicts the result of the heuristics for $q = 1.2$. It was finally chosen as the base for routing of the proposed urban bus network.

The reached reduction is about 23%. This result was preferred by the urban authority, although for $q = 1.5$ it was possible to reduce the length of the original network by 48%. The last mentioned density seemed too small.

8. Conclusion

The paper started with identification and formulation of a problem of road network reduction and its applications to different practical networks was outlined. An exact method of solution based on depth-first search technique was presented and, since the problem is NP-hard, a heuristic method was proposed. Moreover, it was shown that the problem is solvable by linear programming as well. Afterwards, the results of computational experience are drawn up. Finally, a practical application to the urban transport network in the Slovak town Piestany was presented.

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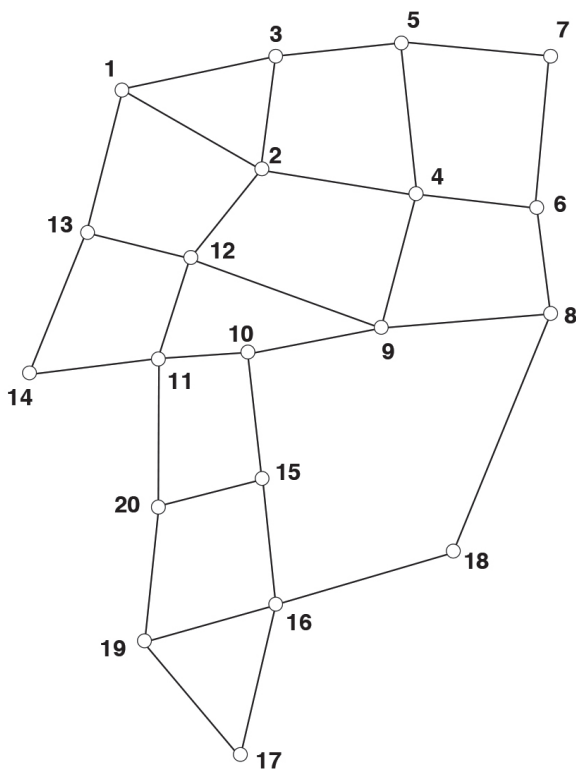


Fig. 6 Piestany - Original network

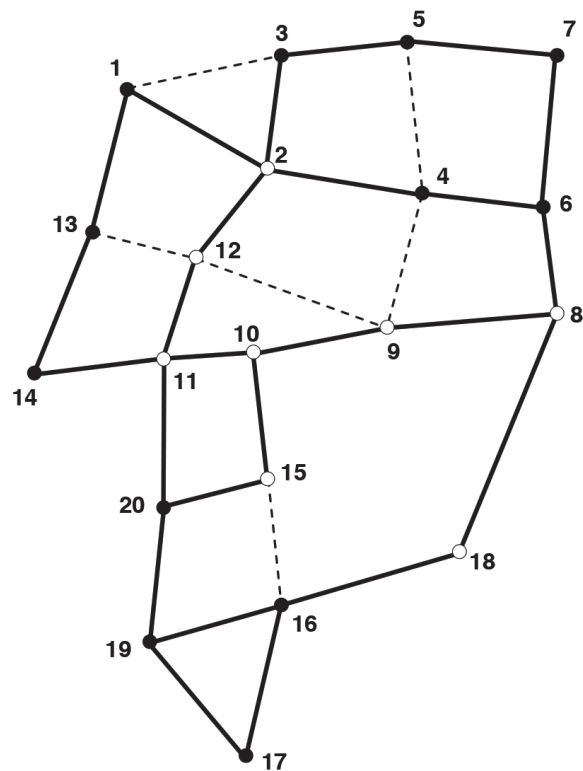


Fig. 7 Piestany - Chosen subnetwork

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