Ludovit Tomanek - Anna Tomankova *

GROUPS WITH THE INFINITE NON-QUASICENTRAL NODAL SUBGROUP

We present some properties of the groups with the infinite non-quasicentral periodic nodal subgroup. Our main results are formulated in Theorem 1, 2 and Theorem 3, 4. 2000 Mathematics Subject Classification: 20F22, 20F25

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The description of groups defined by the systems of their subgroups was first described in the papers of Chernikov and Kurosh (RN – groups, [1]). Chernikov dealt with an extension of the direct product of the finite number of the quasicyclic groups by the finite abelian group (ч – groups, [2]), with the infinite non-abelian groups whose arbitrary infinite subgroup is the normal subgroup of the whole group (INH – groups, [2]). Subbotin studied the groups G in which every subgroup from commutator G´ is a normal subgroup of G (KI–groups, [3]). Tomanek, L. studied the IAN and the IANA groups, Definition 1, (IAN groups, [4]). This definition was given to the author by Chernikov. In this paper we describe IAN and IANA groups with the infinite non-quasicentral periodic nodal subgroup.

We use standard designations of terminology where: $M \times N$ is the direct product of the groups M, N; $\sum_{i \in I} Xi$ is the direct sum of the additive groups X_i for all $i \in I$, $M \rtimes N$ is the semi-direct product of the groups M, N; $M.N = \{mn \mid m \in M, n \in N\}$ is the product of the groups M, N, G/A is the factor group of G by A, $|G{:}N|$ is the index of the subgroup N in a group G; $\langle a \rangle$ is the cyclic group generated by the element a; $\langle a, b, c \rangle$ is the group generated by the elements a, b, c; $H \leq G$ where H is the subgroup of G; $H \trianglelefteq G$, H is normal in G; $[a,b] = a^{-1} b^{-1} ab$ is the commutator of the elements $a,b \in G$; $G' = [G,G]$ generated by all commutators of the elements $a,b \in G$ is the commutator of G; [5] $Z(p^{\infty}) = \{x \mid x^{p^n} = 1, n = 1, 2, ...\}$ is the p-quasicyclic group; $C_G(A)$ is the centralizer of the subgroup A in G; $C(G)$ is the centre of the group G; $G \cong H$ where the groups G, H are isomorphic. The group G is the p- group if each of its elements has an order with a power of some fixed prime p [6]. The group G is a locally graded group if every finitely generated nontrivial subgroup of G contains a proper subgroup of finite index ([2] p.236). The group G is the solvable group if it includes series: $G > G' > G^{(2)} > ... > G^{(n)} = \langle e \rangle$. The subgroup A is a quasicentral of G if every subgroup of A is normal in G. The group G is the extension of the group H by a normal subgroup N of G if $G/N \cong H$. The extension of the finite direct product of the quasicyclic groups by the finite abelian group are ч – groups ([2]). The group G is an almost quasicyclic group if G is the extension of the quasicyclic group by the finite group. $Q_8 = \langle x, y \mid x^4 = y^4 = 1, x^2 = y^2, yx = x^{-1}y \rangle$ is the quaternion group.

Definition 1.

An infinite non-abelian G *is said to be the IAN group if there exists a subgroup A of G so that every infinite subgroup of A and every infinite subgroup of G containing A is a normal subgroup of G. The group G is the IANA group if A is the abelian subgroup. The subgroup A is called the nodal subgroup.*

Definition 2.

An infinite non-abelian G is the INH group if an arbitrary infinite subgroup of G is the normal subgroup of G.

Definition 3.

The group G is the Dedekind group if an arbitrary subgroup of G is the normal subgroup of G. Non-abelian Dedekind group G is called the Hamiltonian group.

Proposition 1. [[2], T. 6.10]

The infinite Hamiltonian groups and the non-abelian non-Hamiltonian groups that are the finite extensions of the quasicyclic

* [1]**Ludovit Tomanek,** [2]**Anna Tomankova**
[1]Department of Mathematics, Faculty of Humanities, University of Zilina, Slovakia
[2]Institute of Foreign Languages, University of Zilina, Slovakia
E-mail: ludovit.tomanek@fhv.uniza.sk

subgroups by the finite abelian and the finite Hamiltonian groups form the class of the solvable INH groups.

Proposition 2.[[7], T.12.5.4]

The group G is the Hamiltonian group if and only if the group $G = Q_8 \times M \times N$ *where* Q_8 *is the quaternion group, M is an elementary abelian 2- group, N is a periodic abelian group with no elements of the order 2.*

Lemma 1.

Let G be the IAN group with a nodal subgroup A. If a nodal subgroup A contains the element of the infinite order, then A is the abelian quasicentral subgroup of group G.

Proof. If the group A contains the element x of the infinite order, then according to Definition 2, the group A is the *INH* group. According to Proposition 1 A is the abelian group. Let B be an arbitrary subgroup of the group A. We shall show that $B \trianglelefteq G$. If B is an infinite subgroup of G, B is admittedly a normal subgroup of G.

Let B be a finite subgroup of G. If A is the abelian group containing the element x of the infinite order, then $B\langle x\rangle \cong B \times \langle x\rangle$. Pursuant to Definition 1 $(B\times\langle x\rangle) \trianglelefteq G$ which implies $B \trianglelefteq G$. Thus A is the abelian quasicentral subgroup of the group G. ■.

Lemma 2.

If G is the locally graded IAN group with the nodal subgroup A, there exists a subgroup of A that is not a normal subgroup of G. Then A is a finite group or A is the extension of the quasicyclic subgroup by the finite Dedekind group.

Proof. Let G be the *IAN* group with a nodal subgroup A and let $A_1 < A$ where A_1 is not a normal subgroup of G. Admittedly, $A_1 \neq \langle e\rangle$ is a finite subgroup of G. In agreement with Lemma 1 A is a periodic group. If A is a finite group, then this lemma is valid. Let A be an infinite periodic subgroup of G. We consider two possible cases; A is not ч - group , or A is ч - group.

Case 1. Let A not be ч - group. Then choose the subgroup A_2 of A where $A_2 = A_3 \times A_4$, $A_2 \cap A_1 = \langle e\rangle$, and where A_3, A_4 are the infinite cyclic groups of G. By Definition 1 $A_3 \trianglelefteq G$, $A_3 \times A_1 \triangleleft G$, $A_4 \trianglelefteq G$, $A_4 \times A_1 \trianglelefteq G$. Evidently $(A_3\times A_1)\cap(A_4\times A_1) = A_1$ and furthermore $A_1 \trianglelefteq G$, so it is a contradiction.

Case 2. Let A be ч- group. Then put $A = R.B$ where R is the direct product of the finite number of the quasicyclic groups, R is at the same time a divisible group, and B is the finite group where $B \neq \langle e\rangle$.Therefore, A_1 is not a normal subgroup of G; there exists a cyclic subgroup $\langle a\rangle$ of A_1 that is not normal in G and where $R \cap \langle a\rangle = \langle t\rangle$. Since R is a divisible group, there exists a quasicyclic subgroup R_1 of R and furthermore R_1 contains the subgroup $\langle t\rangle$.Put $R = R_1 \times R_2$ where R_2 is an infinite subgroup of A or $R_2 = \langle e\rangle$. If R_2 is an infinite subgroup of A, then, by Definition 1 $R_2 \trianglelefteq G$, furthermore $(R_2 \leftthreetimes \langle a\rangle) \trianglelefteq G$, $R_1 \trianglelefteq G$, $(R_1 \langle a\rangle) \trianglelefteq G$. Evidently $(R_2 \leftthreetimes \langle a\rangle) \cap (R_1 \langle a\rangle) = \langle a\rangle$ and $\langle a\rangle \trianglelefteq G$. This is a contradiction.

Let $R_2 = \langle e\rangle$, then $R = R_1$ is a quasicyclic group and moreover $A/R \cong B$ where B is a finite Dedekind group. Thus A is the extension of the quasicyclic subgroup by the finite Dedekind group. ■.

Theorem 1.

If G is a locally graded IAN group with a nodal subgroup A, then subgroup A belongs to one of the types:

1. *A is a finite subgroup of G;*
2. *A is an extension of the quasicyclic subgroup by a finite Dedekind group where G´is an infinite group;*
3. *A is an infinite quasicentral periodic subgroup of G;*
4. *A is a quasicentral non-periodic abelian subgroup of G.*

Proof. If A is not a quasicentral subgroup of G, then, based on Lemma 2, the subgroup A belongs to one of types 1 or 2 of this theorem. If A is a quasicentral subgroup of G, then by Lemma 1 the subgroup A belongs to one of the types 3 or 4 of this theorem. ■.

By Theorem 1 and according to the definition of IANA groups the next corollary follows.

Corollary 1.

If G is a locally graded IANA group with a nodal subgroup A, then subgroup A belongs to one of the types:

1. *A is a finite abelian subgroup of G;*
2. $A = Z(p\infty) \times B$*, where B is a finite group;*
3. *A is an infinite quasicentral periodic abelian subgroup of G;*
4. *A is a quasicentral non-periodic abelian subgroup of G.*

Lemma 3.

If G is the locally graded group with the infinite periodic nodal subgroup A, then the subgroup A satisfies one of the following conditions:

1. *A is the infinite periodic Dedekind quasicentral subgroup of the group G where G/A is the abelian group;*
2. *A is the infinite periodic Dedekind quasicentral subgroup of the group G where G/A is the Hamiltonian group and G is a locally finite group;*
3. *A is not the quasicentral subgroup of G, A is an almost quasicyclic subgroup of G where G/A is the Dedekind group.*

Proof. If G is the locally graded group with an infinite periodic nodal subgroup A, then, according to Theorem 1 A is the extension of the quasicyclic subgroup by the finite Dedekind group, or A is the quasicentral subgroup of the group G.

Let A be the extension of the quasicyclic subgroup B by the finite Dedekind group. If B is an infinite subgroup of G containing A, then $B \trianglelefteq G$ and furthermore every quotient subgroup

$B/A \trianglelefteq G/A$. Since G/A is the Dedekind group, A then satisfies the 3rd condition of this lemma.

If A is a quasicentral subgroup of group G, then, analogous to the paragraph above, we can prove that G/A is the Dedekind group. Admittedly, G/A is the abelian or the Hamiltonian group. If G/A is the abelian group, then A satisfies the 1st condition of this lemma.

Let G/A be the Hamiltonian group. By Proposition 2 G/A is a locally finite group. Thus an extension of a locally finite group by a locally finite group is a locally finite group, which implies that G is a locally finite group and hence A satisfies the 2nd condition of this lemma. ■.

Lemma 4.

If G is the locally graded IAN group with the infinite nodal subgroup A non-quasicentral of G, then A is the extension of a quasicyclic group by the Dedekind group.

Proof. Let G be the locally graded *IAN* group with the infinite nodal subgroup A non-quasicentral of G. According to Theorem 1 A is a periodic group, by Lemma 3 A is the extension of a quasicyclic group by the Dedekind group. ■

Lemma 5.

If G is the group with a finite nodal subgroup A, then G/A is the abelian group, or the group.

Proof. If G/A is the abelian group, then this lemma is valid. Let G/A be a non-abelian group and B/A be an arbitrary infinite subgroup of G/A. There evidently exists $B \trianglelefteq G$ and furthermore $B/A \trianglelefteq G/A$. Thus G/A is the *INH* group. ■

Theorem 2.

Let G be the locally graded IAN group with a nodal subgroup A. The nodal subgroup A of G is a non-quasicentral of G if and only if it satisfies one of the following conditions:

1. *A is a finite non-quasicentral subgroup of G, the quotient group G/A is the INH group with the abelian commutator or G/A is the abelian group;*
2. *A is an almost quasicyclic group which contains the finite subgroups that are not normal in G, $|A: A\cap G| < \infty$, and G/A is the Dedekind group.*

Proof. Let G be the locally graded *IAN* group with the infinite nodal subgroup A. Admittedly, the subgroup A is a non-quasicentral of G. Referring to Lemma 2 A is a finite group, or A is an almost quasicyclic group.

If A is a finite group then, by Lemma 5, G/A is the abelian group or G/A is a solvable *INH* group. According to Proposition 1 the commutator of a solvable *INH* group is the abelian group. Thus the subgroup A satisfies the 1st condition of this theorem.

Let G/A be a solvable *INH* group. By the condition 3 of Lemma 3 G/A A is the Dedekind group. Based on this fact A is an almost quasicyclic group and by Definition 1 A is a non-quasicentral of G. Suppose there exists a finite subgroup and A is normal in G. Therefore G/A is the Dedekind group, A is an almost quasicyclic group, thus G' is a subgroup of that almost quasicyclic subgroup A. G' of G'. Let G' be a finite group and put $A=G'$. Hence A satisfies either condition 1 or condition 2 of this theorem.

If G' is an infinite group, then $|A.G':G'|<\infty$, $|A:AG'|<\infty$, too. Admittedly, A satisfies the 2nd condition of this theorem.

Conversely. Suppose the nodal subgroup A satisfies either condition 1 or condition 2 of this theorem, then G is the *IAN* group with the non-quasicentral nodal subgroup A. ■

By Theorem 2 and the definition of IANA groups the next corollary follows.

Corollary 2.

Let G be a locally graded IANA group with a nodal subgroup A. The nodal subgroup A of G is a non-quasicentral of G if and only if it satisfies one of the following conditions:

1. *A is the finite abelian non-quasicentral subgroup of G, the quotient group G/A is the INH group with the abelian commutator or G/A is the abelian group,*
2. *$A = Z(p\infty)\times D$ where D is the finite abelian subgroup of G, A contains finite subgroups that are not normal in G, and G/A is the Dedekind group.*

Lemma 6.

Let G be the IAN group with the infinite non-quasicentral Dedekind nodal subgroup A. Then $A=Z(p\infty)\times D$ where D is a finite Dedekind group, $p \mid |D|$, and there exists the element $a\in A$ so that the subgroup $\langle a\rangle$ is not normal p - subgroup of G. For $p = 2$ is $D' = \langle e\rangle$.

Proof. Let G be the *IAN* group with the infinite non-quasicentral Dedekind nodal subgroup A. By Lemma 1 A is an almost quasicyclic group. Pursuant to Proposition 2 $A=Z(p\infty)\times D$ where D is a finite Dedekind group, p=2, and $D'=\langle e\rangle$. We shall prove that A does not contain a normal p-group $\langle a\rangle$ of G.

Obviously, if every cyclic subgroup of A is a normal subgroup of G, then A is a quasicentral subgroup of G. Thus there exists a cyclic subgroup $\langle x\rangle$ of A that is not normal in G, which implies that the $\langle a\rangle$ Sylow q- subgroup of the group $\langle x\rangle$ is not normal in G. This verifies that $q = p$.

Let $q \neq p$ and $Z(p\infty)\cap\langle a\rangle = \langle e\rangle$. Then $Z(p\infty).\langle a\rangle \cong Z(p\infty)\times\langle a\rangle$ where $Z(p\infty)\times\langle a\rangle$ is the normal subgroup of G. Evidently, $\langle a\rangle$ is the Sylow q-subgroup normal in $Z(p\infty)\times\langle a\rangle$ and $\langle a\rangle$ is normal in G. This is a contradiction, thus $q = p$.

If $A = Z(p\infty) \times D \geq Z(p\infty)\langle a\rangle$, then $Z(p\infty)\langle a\rangle = Z(p\infty)\times\langle b\rangle$, $\langle b\rangle = D\cap Z(p\infty)\times\langle a\rangle$. Because the subgroup $\langle a\rangle$ is the p- subgroup normal in *G* and furthermore $Z(p^{\infty}) \cap \langle a \rangle < \langle a \rangle$, $|b| > 1$, $\langle b \rangle$ is a *p*-group, therefore $p \mid |D|$. ■

According to Lemma 6 and the Definition of IANA groups, the next corollary follows.

Corollary 3.

Let G be the IANA group with the infinite non-quasicentral nodal subgroup A. Then $A = Z(p^\infty) \times D$, *where D is a finite group,* $p \mid |D|$, *the subgroup D contains an element a A so that the* $\langle a \rangle$ *p - subgroup is not normal in G.*

Theorem 3.

The group G is the locally graded IAN group with the infinite non- quasicentral nodal subgroup A of G if and only if a quotient group G/A is the Dedekind group, $|A : A \cap G'| <$ *and the nodal subgroup A is of one of the types:*

1. $A = Z(p^\infty) \times D$, *where D is the finite Dedekind group,* $p=2$, $D' = \langle e \rangle$, $p \mid |D|$, *the subgroup A contains an element a such that* $\langle a \rangle$ *p-subgroup is not normal in G, and the quotient group* $A/Z(p^\infty)$ *is the quasicentral in* $G/Z(p^\infty)$;
2. $A = Z(p^\infty) \leftthreetimes D$, *where D is the finite Dedekind subgroup, the group A does not contain the finite normal subgroup of G, and* $A/Z(p^\infty)$ *is the quasicentral in* $G/Z(p^\infty)$;
3. $A = (Z(p^\infty).B) \times D$, *where* $Z(p^\infty).B$ *is the non-abelian Syllow p-subgroup of G, D is the infinite Dedekind group,* $p=2$, $D' = \langle e \rangle$, $Z(p^\infty) \leq C(G)$, *the finite group B has a normal series:* $Z(p^\infty) \cap B = B' = B_0 < B_1 < \ldots < B_i < \ldots < B_n$, $n \geq 1$, $B_i = B_{i-1} \leftthreetimes \langle b_i \rangle$, *for all* $i \geq 1$, $|b_i| > 1$, *and* $A/Z(p^\infty)$ *is the quasicentral in* $G/Z(p^\infty)$;
4. $A = (Z(p^\infty).B) \leftthreetimes Q_8) \times D$, *where* $Z(p^\infty).B.Q_8$ *is the Sylow 2-subgroup of G, D is the finite abelian group,* $Z(p^\infty) < C(G)$, *the finite group B has a normal series:* $Z(p^\infty) \cap B = \langle B', [B, Q_8] \rangle = B_0 < B_1 < \ldots < B_i < \ldots < B_n$, $n \geq 1$, $B_i = B_{i-1} \leftthreetimes \langle b_i \rangle$, *for all* $i \geq 1$, $|b_i| = 2$, $|B_0| = 2$, *and* $A/Z(p^\infty)$ *is the quasicentral in* $G/Z(p^\infty)$;
5. $A = (Z(2^\infty).B).\langle d \rangle) \times D$, *where* $Z(2^\infty).B.\langle d \rangle$ *is the Sylow 2 - subgroup of G, D is the finite abelian group,* $Z(2^\infty) \leq C(Z(2^\infty).B)$, *for each* $c \in Z(2^\infty)$, $d^{-1}cd = c^{-1}$, *the finite group B has a normal series :* $Z(2^\infty) \cap B = \langle B', [B, \langle d \rangle] \rangle = B_0 < B_1 < B_i < \ldots < B_n$, $n \geq 1$, $B_i = B_{i-1} \leftthreetimes \langle d \rangle$, *for all* $i \geq 1$, $(Z(p^\infty).B) \cap \langle d \rangle = Z(2^\infty) \cap \langle d \rangle \leq \langle c_1 \rangle$, $|c_1| = 2$, *and* $A/Z(p^\infty)$ *is the quasicentral in* $G/Z(p^\infty)$;
6. $A = (((Z(2^\infty).B).\langle d \rangle) \leftthreetimes Q_8) \times D$, *where* $Z(2^\infty).B.Q_8.\langle d \rangle$ *is the Sylow 2- subgroup of G, D is the finite abelian group,* $Z(2^\infty) < C(Z(2^\infty).B.Q_8)$, *for each* $c \in Z(2^\infty)$, $d^{-1}cd = c^{-1}$, *the finite group B has a normal series:* $Z(2^\infty) \leftthreetimes B = \langle B', [B, \langle d \rangle], [B, Q_8], [\langle d \rangle, Q_8] \rangle = B_0 < B_1 < \ldots < B_i < \ldots < B_n$, $n \geq 1$, $B_i = B_{i-1} \leftthreetimes \langle b_i \rangle$, *for all* $i \geq 1$, $|b_1| = 2$, $|B_0| = 2$, $|d| = 4$, $(Z(p^\infty).B) \cap \langle d \rangle = Z(2^\infty) \cap \langle d \rangle$, *and* $A/Z(p^\infty)$ *is the quasicentral in* $G/Z(p^\infty)$;
7. $A = ((((Z(2^\infty).B) \leftthreetimes Q_8) \times D.B) \leftthreetimes Q_8) \times D$, *where* $Z(2^\infty).B.Q_8$ *is the Sylow 2- subgroup of G, D is the finite abelian group,* $Z(2^\infty) \leq C(Z(2^\infty).B)$, $[Z(2^\infty), Q_8] = Z(2^\infty)$, *the finite group B has a normal series:* $Z(2^\infty).B = \langle B', [B, Q_8] \rangle = B_0 < B_1 < \ldots < B_i < \ldots < B_n$, $n \geq 1$, $B_i = B_{i-1}) \leftthreetimes \langle b_i \rangle$, *for all* $i \geq 1$, $|b_i| = 2$, $|B_0| = 2$ *and* $A/Z(p^\infty)$ *is the quasicentral in* $G/Z(p^\infty)$;
8. $A = ((Z(2^\infty).B).H) \times D$, *where* $Z(2^\infty).B.H$ *is the Sylow 2-subgroup of G, D is the finite abelian group,* $Z(2^\infty) \leq C(Z(2^\infty).B)$, $[Z(2^\infty), H] = Z(2^\infty)$, $H = \langle a \rangle \leftthreetimes \langle b \rangle$, $|a| = |b| = 4$, $[a,b] = a^2$, *the finite group B has a normal series:* $Z(2^\infty) \cap B = \langle B', [B,H] \rangle = B_0 < B_1 < \ldots < B_i < \ldots < B_n$, $n \geq 1$, $B_i = B_{i-1} \leftthreetimes \langle b_i \rangle$, *for all* $i \geq 1$, $|b_1| = 2$, $|B_0| = 2$, $(Z(2^\infty).B) \cap H = Z(2^\infty) \cap H = \langle a^2, b^2 \rangle$, *and* $A/Z(p^\infty)$ *is a quasicentral in* $G/Z(p^\infty)$.

Proof. Let *G* be the locally graded *IAN* group with the infinite non-quasicentral nodal subgroup *A* of *G*. By Theorem 1 *A* is an almost quasicyclic group containing the finite subgroups that are not normal in *G*, $|A : A \cap G| < \infty$, and *G/A* is the Dedekind group. The above mentioned implies that A contains a subgroup $Z(p^\infty)$ that is normal in *G*, $A/Z(p^\infty)$ is the finite Dedekind group and furthermore $Z(p^\infty) \leq B \leq A$. Pursuant to Definition 1 the group *B* is the infinite subgroup of *G*. Admittedly, *B* is normal in *G* and the factor group $A/Z(p^\infty)$ is the quasicentral subgroup of $G/Z(p^\infty)$. According to Theorem 3.1 [8] the subgroup *A* satisfies the conditions of this theorem. Evidently, *A* is the group of one of types 1 to 8 of this theorem.

If *A* is a group of the type 1 of Theorem 3.1 [8], then *A* is the Dedekind group, $A = Z(p^\infty) \times D$ where *D* is the finite Dedekind group, $p = 2$, and $D' = \langle e \rangle$. By Lemma 6 $p \mid |D|$, the subgroup *A* contains element *a* so that a subgroup $\langle a \rangle$ is not normal *p*- subgroup in G . Thus A is of the type 1 of this theorem.

If *A* is a group of one of the types 2 – 8 of Theorem 3.1 [8], then *A* is a subgroup of one of the types 2 - 8 of this theorem .

Conversely. If *G* is a group with the normal subgroup *A* of one of the types 1 – 8 of this theorem , then *G/A* is the Dedekind group. *G* is evidently the locally graded group. Because G/A is the Dedekind group and $A/Z(p^\infty)$ is the quasicentral subgroup of $G/Z(p^\infty)$, then any infinite subgroup contained in *A* and any subgroup which contains a subgroup *A* is normal in *G*. Thus *G* is the *IAN* group.

Let *A* be an infinite subgroup of *G*. If the subgroup *A* is of the type 1 of this theorem, then the subgroup *A* contains a subgroup $\langle a \rangle$ that is not normal in *G*. Thus the subgroup *A* is non-quasicentral subgroup of *G*.

Thus the quasicentral subgroups of the group *G* are the Dedekind groups, which implies *A* is a group of one of the types 2 – 8 of this theorem. Thus *A* is the non-Dedekind group, which implies that the subgroup *A* of one of the types 2 – 8 is a non-quasicentral subgroup of *G*. ■

Theorem 4.

The group G is the IANA group with an infinite non-quasicentral nodal subgroup A, if and only if $A=Z(p^\infty)\times D$, where D is a finite group, $p \mid |D|$, the subgroup A contains an element a so that $\langle a\rangle$ p- subgroup is not normal of G, and $A/Z(p^\infty)$ is the quasicentral in $G/Z(p^\infty)$.

Proof. Let G be the locally graded *IAN* group with the infinite non-quasicentral nodal subgroup A of G, and $A'=\langle e\rangle$. Because G/A is the Dedekind group, $A=Z(p)\times D$, where D is the finite abelian group, A contains the finite subgroups that are not normal in G, and G/A is the Dedekind group. The group G is evidently the locally graded *IAN* group with the nodal subgroup A of the type 1 of Theorem 2, $p \mid |D|$, the subgroup A contains an element a so that $\langle a\rangle$ p - subgroup is not normal in G, and $A/Z(p^\infty)$ is the quasicentral in $G/Z(p^\infty)$.

Let G be a group, A is a subgroup A of G, and $A= Z(p^\infty)\times D$, where D is a finite group, $p \mid |D|$, the subgroup A contains an element a so that $\langle a\rangle$ p - subgroup is not normal in G, and $A/Z(p^\infty)$ is the quasicentral in $G/Z(p^\infty)$.

Conversely. Suppose that $A \trianglelefteq G$ where A is an almost quasicyclic group. Since $A/Z(p^\infty)$ is a quasicentral in $G/Z(p^\infty)$, then $B/Z(p^\infty)\trianglelefteq G/Z(p^\infty)$ for all $B/Z(p^\infty)<A/Z(p^\infty)$. Hence

$B\trianglelefteq G$, A is the abelian subgroup, every infinite subgroup of A and every infinite subgroup of G containing A is a normal subgroup of G. By Definition 1 the group G is the *IANA* group. Hence the subgroup A contains the subgroup that is not normal in G, then A is the non-quasicentral in G. ■

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