Michaela Holesova *

# MONOMIAL CURVES IN AFINNE SPACE AND THEIR ASSOCIATED PRIME IDEALS WITH SIX GENERATORS AS SET-THEORETIC COMPLETE INTERSECTIONS

*The paper deals with the problem of the expression of associated prime ideals of monomial curves in the affine space $A^4$ as set-theoretic complete intersections. We describe some associated prime ideals, a minimal generating set of which has six elements and we prove that these ideals are set-theoretic complete intersections. Corresponding monomial curves are intersections of three hypersurfaces and we find the equations of these hypersurfaces.*

***Keywords:*** *A monomial curve, an associated prime ideal, a set-theoretic complete intersection.*

## 1. Introduction

It is known that $k$-dimensional algebraic affine variety is intersection of not fewer than $n - k$ hypersurfaces in $n$-dimensional affine space $A^n$. There is the presumption, that a number of these hypersurfaces is exactly $n - k$. In this case we can say, that they are ideal-theoretic or set-theoretic complete intersections. This is also equivalent to the fact, that either the associated ideal $I$ of this variety has generators (ideal-theoretic complete intersection) or the ideal $I$ is radical of an ideal $a$, $a \subseteq I$, the ideal $a$ has $n - k$ generators (set-theoretic complete intersection). The number $n - k$ is also height of the ideal $I$. The ideal is called a set-theoretic complete intersection (s.t.c.i., for short), if there are $s$=ht($I$) elements $g_1, g_2, g_3, \ldots g_s$, such that rad($I$)=rad($g_1, g_2, g_3, \ldots g_s$).

Let $K$ be an arbitrary field, $R = K[x_1, x_2, x_3, x_4,]$ the polynomial ring in four variables over $K. C = C(n_1, n_2, n_3, n_4)$ a monomial curve in affine space $A^4$ over $K$ having parameterization $x_i = t^{n_i}, i \in \{1,2,3,4\}$, where $n_1, n_2, n_3, n_4$ be positive integers with g.c.d. equal 1 and $n_1, n_2, n_3, n_4$ is a minimal set of generators for the numerical semigroup $H = \langle n_1, n_2, n_3, n_4 \rangle$.

The ideal $P$ of all polynomials $f(x_1, x_2, x_3, x_4) \in R$ such that $f(t^{n_1}, t^{n_2}, t^{n_3}, t^{n_4}) = 0$, $t$ transcendental over $K$, is the associated prime ideal of ring $R$ of the monomial curve $C$. $P$ is the corresponding ideal with dim($P$)=1 and height ht($P$)=3. In particular, associated prime ideal $P$ of monomial curve $C$ in $A^4$ is a s.t.c.i., if $P$=rad($g_1, g_2, g_3$) and also a monomial curve $C$ is a s.t.c.i. (more information in [1]).

The general problem of whether all associated ideals of monomial curves (or monomial curves) are s.t.c.i. is still open. There are nevertheless some partial results in this direction. E.Kunz [1] showed that every monomial curve in 3-dimensional affine space is a s.t.c.i.

In 4-dimensional affine space $A^4$, H. Bresinsky proved that if numerical semigroup $H$ is symmetric, then the monomial curve $C(n_1, n_2, n_3, n_4)$ and its associated prime ideals are s.t.c.i.(see [2]). D. Patil presented in [3], if $n - 1$ numbers among $m_1, m_1, \ldots, m_n$ form an arithmetic sequence, then $C$=$C(m_1, m_1, \ldots, m_n)$ in $A^n$ is s.t.c.i. S.Solcan dealt with monomial curves $C(p^2, p^2+p, p^2+p+1, (p+1)^2)$ in $A^4$ as s.t.c.i. for the positive characteristic of the field $K$ char $K = p \neq 0$[4] and for char $K$ = 0 [5]. W. Gastinger in [6] proved that associated prime ideals of monomial curves in $A^4$ are s.t.c.i. if minimal generating sets of these ideals have four generators. We showed that associated prime ideals of monomial curves whose minimal set of generators have five elements is s.t.c.i. [7].

## 2. The associated prime ideal *P* of the monomial curve *C*

Let a binomial term $\prod_{i=1}^{4} x_i^{\gamma_i} - \prod_{i=1}^{4} x_i^{\vartheta_i} \in P$, where $\gamma_i \vartheta_i = 0,\ i \in \{1,2,3,4\}$. It is clear that $\sum_{i=1}^{4} \gamma_i n_i = \sum_{i=1}^{4} \vartheta_i n_i$.

We have basically two types of binomial terms of $P$:

$x_i^{\gamma_i} x_j^{\gamma_j} - x_k^{\gamma_k} x_l^{\gamma_l}, \{i,j,k,l\} = \{1,2,3,4\}, \gamma_i \gamma_j \gamma_k \gamma_l \neq 0$

or $x_i^{r_i} - x_j^{\alpha_{ij}} x_k^{\alpha_{ik}} x_l^{\alpha_{il}}, \{i,j,k,l\} = \{1,2,3,4\}, r_i \neq 0$.

We denote the binomial term $x_i^{r_i} - x_j^{\alpha_{ij}} x_k^{\alpha_{ik}} x_l^{\alpha_{il}}$ by $(x_i^{r_i})$ if $r_i$ is minimal and by $(x_i^{r_i}, x_j^{r_j})$ if $x_j^{r_j} - x_i^{\alpha_{ji}} x_k^{\alpha_{jk}} x_l^{\alpha_{jl}} \in P$ with $r_j$ minimal and $\alpha_{ji} = r_i, \alpha_{jk} = \alpha_{jl} = 0$. Every generating set for

* **Michaela Holesova**
Department of Mathematics, Faculty of Humanities, University of Zilina, Slovakia
E-mail: michaela.holesova@fhv.uniza.sk

$P$ contains for each $i\left(i \in \{1,2,3,4\}\right)$ at least one polynomial $x_i^{r_i} - x_j^{\alpha_{ij}} x_k^{\alpha_{ik}} x_l^{\alpha_{il}}$ with $r_i$ minimal. We also denote polynomial $x_i^{r_i'} - x_j^{\alpha_{ij}} x_k^{\alpha_{ik}} x_l^{\alpha_{il}}$ by $\left(x_i^{r_i'}(k,l)\right)$ if $r_i'$ is minimal with respect to the condition either $\alpha_{ik} \neq 0$ or $\alpha_{il} \neq 0$. We defined as H. Bresinsky a set in three cases as follows:

- For binomials $\left(x_s^{r_s}\right)$, $s = i,j,k,l,\{i,j,k,l\}$ with at least two exponents $\alpha_{sh}$ not zero, $h \in \{i,j,k,l\} - \{s\}$ let $B = \left\{\left(x_i^{r_i}\right),\left(x_j^{rj}\right),\left(x_k^{r_k}\right),\left(x_l^{r_l}\right)\right\}$
- Let $\left(x_i^{r_i},x_j^{r_j}\right) \in P$ but $\left(x_k^{r_k},x_l^{r_l}\right) \notin P$. Then either $B = \left\{\left(x_i^{r_i},x_j^{r_j}\right),\left(x_k^{r_k}\right),\left(x_l^{r_l}\right)\right\}$ or $B = \left\{\left(x_i^{r_i},x_j^{r_j}\right),\left(x_k^{r_k}\right),\left(x_l^{r_l}\right),\left(x_j^{r_j'}(k,l)\right)\right\}$,
- $B = \left\{\left(x_i^{r_i},x_j^{r_j}\right),\left(x_k^{r_k},x_l^{r_l}\right)\right\} \cup C, C \subseteq \left\{\left(x_j^{r_j'}(k,l)\right),\left(x_l^{r_l'}(i,j)\right)\right\}$.

We write $x_i^{\gamma_{i1}} x_j^{\gamma_{j1}} \not< x_i^{\gamma_{i2}} x_j^{\gamma_{j2}}$ if either $\gamma_{i1} > \gamma_{i2}$ and $\gamma_{j1} > \gamma_{j2}$ or the inequalities are reversed. For binomials we write $x_i^{\gamma_{i1}} x_j^{\gamma_{j1}} - x_k^{\gamma_{k1}} x_l^{\gamma_{l1}} \not< x_l^{\gamma_{i2}} x_j^{\gamma_{j2}} - x_k^{\gamma_{k2}} x_l^{\gamma_{l2}}$ if $\not<$ holds between the first and second monomials of this binomials, i.e. if $x_i^{\gamma_{i1}} x_j^{\gamma_{j1}} \not< x_i^{\gamma_{i2}} x_j^{\gamma_{j2}}$ and $x_k^{\gamma_{k1}} x_l^{\gamma_{l1}} \not< x_k^{\gamma_{k12}} x_l^{\gamma_{l2}}$.

We next define a set $D_{ij}, i \neq j, \{i,j\} \subset \{1,2,3,4\}$,
$D_{i,j} = \{f = x_i^{\gamma_i} x_j^{\gamma_j} - x_k^{\gamma_k} x_l^{\gamma_l}, \{k,l\} \subset \{1,2,3,4\} - \{i,j\}$,
$\gamma_h < r_h'$ for the polynomials $\left(x_h^{r_h'}(k,l)\right)$ if $h \in \{i,j\}$, for the polynomials $\left(x_h^{r_h'}(i,j)\right)$ if $h \in \{k,l\}$ and for each binomial term $f' = x_i^{\gamma_{i1}} x_j^{\gamma_{j1}} - x_k^{\gamma_{k1}} x_l^{\gamma_{l1}} \in P, f' \neq f$ is $f' \not< f\}$.

The definition of sets $B, D_{ij}$ gives

**Corollary 2.1**

1. If $B = \left\{\left(x_1^{r_1}\right),\left(x_2^{r_2}\right),\left(x_3^{r_3}\right),\left(x_4^{r_4}\right)\right\}$ and $x_i^{\gamma_i} x_j^{\gamma_j} - x_k^{\gamma_k} x_l^{\gamma_l} \in D_{i,j}$, $\{i,j,k,l\} = \{1,2,3,4\}$, then $\gamma_h < r_h, h \in \{1,2,3,4\}$.
2. Let $B = \left\{\left(x_i^{r_i},x_j^{r_j}\right),\left(x_k^{r_k}\right),\left(x_l^{r_l}\right)\right\}$. If $x_i^{\gamma_i} x_k^{\gamma_k} - x_j^{\gamma_j} x_l^{\gamma_l} \in D_{ik}$, or $x_i^{\gamma_i} x_l^{\gamma_l} - x_j^{\gamma_j} x_k^{\gamma_k} \in D_{il}$, $\{i,j,k,l\} = \{1,2,3,4\}$, then $\gamma_h < r_h, h \in \{i,j\}$. If $x_i^{\gamma_i} x_j^{\gamma_j} - x_k^{\gamma_k} x_l^{\gamma_l} \in D_{i,j}$, $\{i,j,k,l\} = \{1,2,3,4\}$, then $\gamma_h < r_h, h \in \{k,l\}$.
3. Let $B = \left\{\left(x_i^{r_i},x_j^{r_j}\right),\left(x_k^{r_k}\right),\left(x_l^{r_l}\right),\left(x_j^{r_j'}(k,l)\right)\right\}$. If $x_i^{\gamma_i} x_k^{\gamma_k} - x_j^{\gamma_j} x_l^{\gamma_l} \in D_{ik}$, or $x_i^{\gamma_i} x_l^{\gamma_l} - x_j^{\gamma_j} x_k^{\gamma_k} \in D_{il}$, $\{i,j,k,l\} = \{1,2,3,4\}$, then $\gamma_h < r_h, h \in \{i,j\}$. If $x_i^{\gamma_i} x_j^{\gamma_j} - x_k^{\gamma_k} x_l^{\gamma_l} \in D_{i,j}$, $\{i,j,k,l\} = \{1,2,3,4\}$, then $\gamma_h < r_h, h \in \{k,l\}$.
4. Let $B = \left\{\left(x_i^{r_i},x_j^{r_j}\right),\left(x_k^{r_k},x_l^{r_l}\right)\right\} \cup C, C \subseteq \left\{\left(x_j^{r_j'}(k,l)\right),\left(x_l^{r_l'}(i,j)\right)\right\}$. If $x_i^{\gamma_i} x_k^{\gamma_k} - x_j^{\gamma_j} x_l^{\gamma_l} \in D_{ik}$, or $x_i^{\gamma_i} x_l^{\gamma_l} - x_j^{\gamma_j} x_k^{\gamma_k} \in D_{il}$, $\{i,j,k,l\} = \{1,2,3,4\}$, then $\gamma_h < r_h, h \in \{1,2,3,4\}$. If $x_i^{\gamma_i} x_j^{\gamma_j} - x_k^{\gamma_k} x_l^{\gamma_l} \in D_{i,j}$, $\{i,j,k,l\} = \{1,2,3,4\}$, then $\gamma_h < r_h', h \in \{j,l\}$.

In [8], H. Bresinsky gives the following theorem.

**Theorem 2.1**
*$M = B \cup D_{ij} \cup D_{ik} \cup D_{il}, \{i,j,k,l\} = \{1,2,3,4\}$ is a minimal generating set for the associated prime ideal $P = P(n_1,n_2,n_3,n_4)$ of the monomial curve in $A^4$.*

We can find in [6], Lemma 7.1, the next property of minimal generating set for a prime ideal *P*.

**Lemma 2.1** *Let $g_j = \prod_{i=1}^{4} x_i^{\gamma_i}, j = 1,...,t$ be a monomial term in R.*

*Let $M = \{x_1^{r_1} - g_1, x_2^{r_2} - g_2, g_3 - g_4, ..., g_{t-1} - g_t\}$ be a minimal generating set for the associated prime ideal P of the monomial curve in $A^4$. If $x_2 \mid g_1$ and $x_1 \mid g_2$, then there is an integer $k, 3 \leq k \leq t$ with $g_k = x_1^{\delta_1} x_2^{\delta_2}$.*

Following theorems we proved in [7].

**Theorem 2.2** *Let $B = \left\{\left(x_1^{r_1}\right),\left(x_2^{r_2}\right),\left(x_3^{r_3}\right),\left(x_4^{r_4}\right)\right\} \subseteq M$, where M is a minimal generating set for the associated prime ideal P of a monomial curve in $A^4$. If $x_i^{r_i} - x_j^{\alpha_{ij}} x_k^{\alpha_{ik}} x_l^{\alpha_{il}} \in B$ and $\alpha_{ij} \neq 0$, $\{i,j,k,l\} = \{1,2,3,4\}$, then $\alpha_{ji} < r_i$.*

**Theorem 2.3** *Let $B = \left\{\left(x_i^{r_i},x_j^{r_j}\right),\left(x_k^{r_k}\right),\left(x_l^{r_l}\right),\left(x_j^{r_j'}(k,l)\right)\right\}$ and $B \subseteq M, \{i,j,k,l\} = \{1,2,3,4\}$ where M is a minimal generating set for the associated prime ideal P of a monomial curve in $A^4$.*
*If $\alpha_{kl} \neq 0$, then $\alpha_{lk} < r_k$.*
*If $\alpha_{lk} \neq 0$, then $\alpha_{kl} < r_l$.*
*If $\alpha_{ju} \neq 0$, then $\alpha_{uj} < r_j', u = k,l$.*
*If $\alpha_{uj} \neq 0$, then $\alpha_{ju} < r_u, u = k,l$.*

**Theorem 2.4** *If $B = \left\{\left(x_i^{r_i},x_j^{r_j}\right),\left(x_k^{r_k},x_l^{r_l}\right)\right\} \cup C \subseteq M$, $C \subseteq \left\{\left(x_j^{r_j'}(k,l)\right) = x_j^{r_j'} - x_i^{\alpha_{ji}} x_k^{\alpha_{jk}} x_l^{\alpha_{jl}} \left(x_l^{r_l'}(i,j)\right) = x_l^{r_l'} - x_i^{\alpha_{li}} x_j^{\alpha_{lj}} x_k^{\alpha_{lk}}\right\}$, $\{i,j,k,l\} = \{1,2,3,4\}$ and M is a minimal generating set for the associated prime ideal P of a monomial curve in $A^4$. Then $\alpha_{ji} < r_i$ and $\alpha_{lk} < r_k$.*

**Theorem 2.5** *Let P be the associated prime ideal of a monomial curve in $A^4$.*
*Let $B = \left\{\left(x_1^{r_1}\right),\left(x_2^{r_2}\right),\left(x_3^{r_3}\right),\left(x_4^{r_4}\right)\right\}$ and $M = B \cup D_{12} \cup D_{13} \cup D_{14}$ be a minimal generating set for the ideal P, $x_i^{\gamma_i} x_j^{\gamma_j} - x_k^{\gamma_k} x_l^{\gamma_l} \in D_{ij}, \{i,j,k,l\} = \{1,2,3,4\}$. If $\left(x_s^{r_s}\right) \in B, s \in \{k,l\}, \left(x_s^{r_s}\right) = x_s^{r_s} - N_s$ and $N_s = x_i^{\alpha_{si}} x_j^{\alpha_{sj}}$, then either $\alpha_{si} > \gamma_i \wedge \alpha_{sj} < \gamma_j$ or $\alpha_{si} < \gamma_i \wedge \alpha_{sj} > \gamma_j$.*

When we use permutation of index $(i,j,k,l) \rightarrow (k,l,i,j)$ in Theorem 2.5 we receive following

**Corollary 2.2** *Let P be the associated prime ideal of a monomial curve in $A^4$.*
*Let $B = \left\{\left(x_1^{r_1}\right),\left(x_2^{r_2}\right),\left(x_3^{r_3}\right),\left(x_4^{r_4}\right)\right\}$ and $M = B \cup D_{12} \cup D_{13} \cup D_{14}$ be a minimal generating set for the ideal P, $x_i^{\gamma_i} x_j^{\gamma_j} - x_k^{\gamma_k} x_l^{\gamma_l} \in D_{ij}, \{i,j,k,l\} = \{1,2,3,4\}$. If $\left(x_s^{r_s}\right) \in B, s \in \{k,l\}, \left(x_s^{r_s}\right) = x_s^{r_s} - N_s$ and $N_s = x_k^{\alpha_{sk}} x_l^{\alpha_{sl}}$, then either $\alpha_{sk} > \gamma_k \wedge \alpha_{sl} < \gamma_l$ or $\alpha_{sk} < \gamma_k \wedge \alpha_{sl} > \gamma_l$.*

## 3. The minimal generating set for a prime ideal *P* having six generators

The inducted propositions give necessary conditions on a minimal set of generators for an associated prime ideal *P* of a monomial curve in $A^4$. When we use methods presented in [9]

- [11] and we also suppose that the minimal generating set $M$ has six generators

$$M=\{x_i^{r_i}-x_j^{\alpha_{ij}}x_k^{\alpha_{ik}}x_l^{\alpha_{il}}, x_j^{r_j}-x_i^{\alpha_{ji}}x_k^{\alpha_{jk}}x_l^{\alpha_{jl}},\\ x_k^{r_k}-x_i^{\alpha_{ki}}x_j^{\alpha_{kj}}x_l^{\alpha_{kl}}, x_l^{r_l}-x_i^{\alpha_{li}}, x_i^{\omega_i}x_k^{\omega_k}-x_j^{\omega_j}x_l^{\omega_l},\\ x_i^{\gamma_i}x_l^{\gamma_l}-x_j^{\gamma_j}x_k^{\gamma_k}\} \tag{1}$$

where $\{i,j,k,l\}=\{1,2,3,4\}$, $\alpha_{ik}$ and $\alpha_{jk}$ are not equal to zero, we get that there is an $u\in N_0$ and all exponents must satisfy the following equations:

$$\begin{aligned} &r_i=\alpha_{ji}+\alpha_{ki}+(u+1)\alpha_{li},\\ &r_j=\alpha_{ij}+\alpha_{kj}+(u+1)\alpha_{lj},\\ &r_k=\alpha_{ik}+\alpha_{jk},\\ &r_l=\omega_l+\alpha_{jl}=\gamma_l+\alpha_{il},\\ &(u+1)r_l=\alpha_{il}+\alpha_{jl}+\alpha_{kl},\\ &\omega_i=\alpha_{li}+\alpha_{ji}, \omega_j=\alpha_{ij}+\alpha_{kj}+u\alpha_{lj},\\ &\omega_k=\alpha_{jk}, \omega_l=\alpha_{il}+\alpha_{kl}-ur_l,\\ &\gamma_i=\alpha_{ji}+\alpha_{ki}+u\alpha_{li}, \gamma_j=\alpha_{ij}+\alpha_{lj},\\ &\gamma_k=\alpha_{ik}, \gamma_l=\alpha_{jl}+\alpha_{kl}-ur_l. \end{aligned} \tag{2}$$

## 4. Set-theoretic complete intersection

In this section we will prove that each associated prime ideal $P$ of the monomial curve $C$, whose minimal generating set $M$ has description (1) is s.t.c.i. and the corresponding monomial curve $C$ is also s.t.c.i. We show that $C$ is an intersection of three hypersurfaces and we give the equations of these hypersurfaces.

**Theorem 4.1** *Let $P$ be the associated prime ideal of the monomial curve $C$ in $A^4$.*

*If* $M=\{x_i^{r_i}-x_j^{\alpha_{ij}}x_k^{\alpha_{ik}}x_l^{\alpha_{il}}, x_j^{r_j}-x_i^{\alpha_{jk}}x_k^{\alpha_{jk}}x_l^{\alpha_{jl}}, x_k^{r_k}-x_i^{\alpha_{ki}}x_j^{\alpha_{kj}}x_l^{\alpha_{kl}}, x_l^{r_l}-x_i^{\alpha_{li}}x_j^{\alpha_{lj}}, x_i^{\omega_i}x_k^{\omega_k}-x_j^{\omega_j}x_l^{\omega_l}x_i^{\gamma_i}x_l^{\gamma_l}-x_j^{\gamma_j}x_k^{\gamma_k}\}$ *is a minimal generating set for the prime ideal P, where* $\{i,j,k,l\}=\{1,2,3,4\}$, $\alpha_{ik}$ *and* $\alpha_{jk}$ *are not equal to zero and exponents satisfy equations (2) for some* $u\in N_0$, *then this prime ideal P (monomial curve C) is a set-theoretic complete intersection.*

**Proof.** To prove our claim we need to show that $P=Rad(g_1,g_2,g_3)$, $g_s\in P, s\in\{1,2,3\}$. We find expression of polynomials $g_s\in P, s\in\{1,2,3\}$.

Now we denote polynomials from the minimal generating set $M$ as $g_2=x_k^{r_k}-x_l^{\alpha_{ki}}x_j^{\alpha_{kj}}x_l^{\alpha_{kl}}, g_3=x_l^{r_l}-x_i^{\alpha_{lj}}, \{i,j,k,l\}=\{1,2,3,4\}$. When we add a polynomial $g\in(g_2,g_3)$ to a polynomial $f\in R$ we denote it as $\rightarrow$. Let $F_1=x_j^{r_j}-x_i^{\alpha_{ji}}x_k^{\alpha_{jk}}x_l^{\alpha_{jl}}$, then we have

$$F_1^{r_kr_l}=(-1)^{r_kr_l}\sum_{h=0}^{r_kr_l}r_kr_l\binom{r_kr_l}{h}(-1)^h\left(x_i^{\alpha_{ji}}x_k^{\alpha_{jk}}x_l^{\alpha_{jl}}\right)^{r_kr_l-h}x_j^{r_jh}.$$

We denote $b_h=r_kr_l\binom{r_kr_l}{h}(-1)^{r_kr_l+h}, h=0,...,r_kr_l$.

$$F_1^{r_kr_l}\rightarrow\sum_{h=0}^{\alpha_{jk}r_l}b_hx_i^{\alpha_{ji}(r_kr_l-h)+\alpha_{ki}(\alpha_{jk}r_l-h)}x_k^{\alpha_{ik}h}x_l^{\alpha_{kl}(\alpha_{jk}r_l-h)+\alpha_{jl}(r_kr_l-h)}x_j^{r_jh+\alpha_{kj}(\alpha_{jk}r_l-h)}$$

$$+\sum_{h=\alpha_{jk}r_l+1}^{r_kr_l}b_h\left(x_i^{\alpha_{ji}}x_k^{\alpha_{jk}}x_l^{\alpha_{jl}}\right)^{r_kr_l-h}x_j^{r_jh}$$

[if $h\leq\alpha_{jk}r_l$, then $x_k^{\alpha_{jk}(r_kr_l-h)}\rightarrow x_k^{\alpha_{ik}h}\left(x_i^{\alpha_{ki}}x_j^{\alpha_{kj}}x_l^{\alpha_{kl}}\right)^{(\alpha_{jk}r_l-h)}$]

$$\rightarrow\sum_{h=0}^{\alpha_{jk}r_l}b_h\Big(x_i^{\alpha_{ji}(r_kr_l-h)+\alpha ki(\alpha_{jk}r_l-h)+\alpha_{li}(\alpha_{jl}\alpha_{ik}+u(\alpha_{jk}r_l-h))}x_k^{\alpha_{ik}h}$$
$$.x_l^{\gamma_l(\alpha_{jk}r_l-h)}x_j^{r_jh+\alpha_{kj}(\alpha_{jk}r_l-h)+\alpha_{lj}(\alpha_{jl}\alpha_{ik}+u(\alpha_{jk}r_l-h))}\Big)$$

$$+\sum_{h=\alpha_{jk}r_l+1}^{r_kr_l}b_h\left(x_i^{\alpha_{ji}}x_k^{\alpha_{jk}}x_l^{\alpha_{jl}}\right)^{r_kr_l-h}x_j^{r_jh}$$

[if $h\leq\alpha_{jk}r_l$, then

$x_l^{\alpha_{kl}(\alpha_{jk}r_l-h)+\alpha_{jl}(r_kr_l-h)}\rightarrow x_l^{\gamma_l(\alpha_{jk}r_l-h)}\left(x_i^{\alpha_{li}}x_j^{\alpha_{lj}}\right)^{\alpha_{jl}\alpha_{ik}+u(\alpha_{jk}r_l-h)}$]

$$\rightarrow\sum_{h=0}^{\alpha_{jk}\gamma_l-1}b_h\Big(x_i^{\alpha_{ji}(r_kr_l-h)+\alpha_{ki}(\alpha_{jk}r_l-h)+\alpha_{li}(\alpha_{jl}\alpha_{ik}+u(\alpha_{jk}r_l-h)+\alpha_{jk}\gamma_l-h)}x_k^{\alpha_{ik}h}$$
$$.x_l^{\alpha_{il}h}x_j^{r_jh+\alpha_{kj}(\alpha_{jk}r_l-h)+\alpha_{lj}(\alpha_{jl}\alpha_{ik}+u(\alpha_{jk}r_l-h)+\alpha_{jk}\gamma_l-h)}\Big)$$
$$+\sum_{h=\alpha_{jk}\gamma_l}^{\alpha_{jk}r_l}b_h\Big(x_i^{\alpha_{ji}(r_kr_l-h)+\alpha ki(\alpha_{jk}r_l-h)+\alpha_{li}(\alpha_{jl}\alpha_{ik}+u(\alpha_{jk}r_l-h))}x_k^{\alpha_{ik}h}$$
$$.x_l^{\alpha_{il}h}x_j^{r_jh+\alpha_{kj}(\alpha_{jk}r_l-h)+\alpha_{lj}(\alpha_{jl}\alpha_{ik}+u(\alpha_{jk}r_l-h)+\alpha_{jk}\gamma_l-h)}\Big)$$

[if $h<\alpha_{jk}\gamma_l$, then $x_l^{\gamma_l(\alpha_{jk}r_l-h)}\rightarrow x_l^{\alpha_{il}h}\left(x_i^{\alpha_{li}}x_j^{\alpha_{lj}}\right)^{\alpha_{jk}\gamma_l-h}$]

$$=x_j^{\alpha_{lj}\alpha_{jk}\alpha_{kl}+\alpha_{kj}\alpha_{jk}r_l+\alpha_{lj}\alpha_{jl}r_k}\Big(\sum_{h=0}^{\alpha_{jk}\gamma_l-1}b_h\left(x_j^{\alpha_{ij}}x_k^{\alpha_{ik}}x_l^{\alpha_{il}}\right)^h$$
$$.\,x_i^{\alpha_{ji}r_kr_l+\alpha_{jk}r_l(\alpha_{ki}+u\alpha_{li})+\alpha_{li}(\alpha_{jk}\gamma_l+\alpha_{jl}\alpha_{ik})-r_ih}$$
$$+\sum_{h=\alpha_{jk}\gamma_l}^{\alpha_{jk}r_l}b_hx_i^{\alpha_{ji}r_kr_l+\alpha_{jk}r_l(\alpha_{ki}+u\alpha_{li})+\alpha_{li}\alpha_{jl}\alpha_{ik}-(r_i-\alpha_{li})h}x_k^{\alpha_{ik}h}x_l^{\gamma_l(\alpha_{jk}r_l-h)}x_j^{\alpha_{ij}h+\alpha_{lj}(h-\alpha_{jk}\gamma_l)}$$
$$+\sum_{h=\alpha_{jk}r_l+1}^{r_kr_l}b_h\left(x_i^{\alpha_{ji}}x_k^{\alpha_{jk}}x_l^{\alpha_{jl}}\right)^{r_kr_l-h}x_j^{\alpha_{ij}h+(\alpha_{lj}u+\alpha_{kj})(h-\alpha_{jk}r_l)+\alpha_{lj}(h-\alpha_{jk}\gamma_l-\alpha_{jl}\alpha_{ik})}\Big)$$
$$=x_j^{\alpha_{lj}\alpha_{jk}\alpha_{kl}+\alpha_{kj}\alpha_{jk}r_l+\alpha_{lj}\alpha_{jl}r_k}g_1.$$

Let $F_2=x_i^{r_i}-x_j^{\alpha_{ij}}x_k^{\alpha_{ik}}x_l^{\alpha_{il}}$ be another generator of the ideal $P$. We know that $x_j^{(u+1)\alpha_{lj}+\alpha_{kj}}F_2\equiv-x_k^{\alpha_{ik}}x_l^{\alpha_{il}}F_1\ mod(g_2,g_3)$

and it is easy to see that

$x_j^{((u+1)\alpha_{lj}+\alpha_{kj})r_kr_l}F_2^{r_kr_l}\equiv(-1)^{r_kr_l}\left(x_i^{\alpha_{ki}}x_j^{\alpha_{kj}}\right)^{\alpha_{ik}r_l}\left(x_i^{\alpha_{li}}x_j^{\alpha_{lj}}\right)^{\alpha_{il}r_k}F_1^{r_kr_l}$

$mod(g_2,g_3)$.

and

$$F_1^{r_kr_l}\equiv x_j^{\alpha_{lj}\alpha_{jk}\alpha_{kl}+\alpha_{kj}\alpha_{jk}r_l+\alpha_{lj}\alpha_{jl}r_k}g_1\ mod(g_2,g_3). \tag{3}$$

We know that $R/(g_2,g_3)$ is a module over $K[x_i,x_j]$ and $\{g_2,g_3\}$ is a Grobner basis for $(g_2,g_3)$ with respect to the lexicographic order, taking $x_l>x_k>x_j>x_i$. By [12], Chapter 1, § 3, Exercise 4 each element $\bar{f}\in R/(g_2,g_3)$is uniquely expressed $\bar{f}=a_1^1\cdot1+\dots+a_1^{r_k}\cdot x_k^{r_k-1}+a_2^1\cdot x_1+\dots+a_2^{r_l}\cdot x_lx_k^{r_k-1}+\dots+a_{r_l}^1\cdot x_l^{r_l-1}+\dots+a_{r_l}^{r_l}\cdot x_l^{r_l-1}x_k^{r_k-1}+(g_2,g_3)$, $a_d^n\in K[x_i,x_j], n=1,2,...,r_l$. Clearly, the module $R/(g_2,g_3)$ has a linearly independent basis $\{\bar{1},...,\overline{x_l^{r_l-1}x_k^{r_k-1}}\}$ over $K[x_i,x_j]$, thus is free module over $K[x_l,k_j]$ and its rank is $r_kr_l$. Therefore

$$F_2^{r_kr_l}\equiv(-1)^{r_kr_l}x_j^{\alpha_{ki}\alpha_{ik}r_l+\alpha_{li}(\alpha_{jl}r_k+\alpha_{ik}\alpha_{kl})}g_1\ mod(g_2,g_3). \tag{4}$$

We denote the generator $x_i^{\omega_i}x_k^{\omega_k}-x_j^{\omega_j}x_l^{\omega_l}$ of ideal $P$ as $F_3$. Moreover,

$x_j^{\alpha_{lj}}F_3\equiv-x_l^{\alpha_{il}+\alpha_{kl}-ur_l}F_1\ \ mod(g_2,g_3)$, hence

$x_j^{\alpha_{lj}r_kr_l}F_3^{r_kr_l} \equiv (-1)^{r_kr_l}\left(x_i^{\alpha_{li}}x_j^{\alpha_{lj}}\right)^{r_k(\alpha_{il}+\alpha_{kl}-ur_l)}F_1^{r_kr_l} \; mod(g_2,g_3)$.

We use the same properties as above and we have a consequence,

$$F_3^{r_kr_l} \equiv (-1)^{r_kr_l}x_i^{\alpha_{li}r_k(\alpha_{li}+\gamma_l-\alpha_{il})}x_j^{\alpha_{jk}(\alpha_{kj}r_l+\alpha_{lj}\alpha_{kl})}g_1 \; mod(g_2,g_3) \qquad (5)$$

We denote the generator $x_i^{\gamma_i}x_l^{\gamma_l}-x_j^{\gamma_j}x_k^{\gamma_k}$ of ideal $P$ as $F_4$. We know that $x_j^{\alpha_{kj}+u\alpha_{lj}}F_4 \equiv -x_k^{\alpha_{ik}}F_1 \; mod(g_2,g_3)$ and
$x_j^{(\alpha_{kj}+u\alpha_{lj})r_kr_l}F_4^{r_kr_l} \equiv (-1)^{r_kr_l}\left(x_i^{\alpha_{ki}}x_j^{\alpha_{kj}}\right)^{r_l\alpha_{ik}}\left(x_l^{\alpha_{li}}x_j^{\alpha_{lj}}\right)^{\alpha_{kl}\alpha_{ik}}F_1^{r_kr_l} \; mod(g_2,g_3)$

When we use the same method as above, we get

$$F_4^{r_kr_l} \equiv (-1)^{r_kr_l}x_i^{\alpha_{ki}\alpha_{ik}r_l+\alpha_{li}\alpha_{kl}\alpha_{ik}}x_j^{\alpha_{ljk}\gamma_lr_k}g_1 \; mod(g_2,g_3) \qquad (6)$$

If $P$ is the associated prime ideal of the monomial curve, then $P$=Rad($P$) and $g_1 \in P$. This implies that $(g_1,g_2,g_3) \subseteq P$ and this inclusion induces Rad$(g_1,g_2,g_3) \subseteq P =$ Rad($P$). From (3), (4), (5), (6) then we can easily get $P \subseteq$Rad$(g_1,g_2,g_3)$. Then $P$=Rad$(g_1,g_2,g_3)$ and the proof is completed.

**Example 4.1.** We take the minimal generating set $M_t$ for the associated prime ideals $P_t, t \in \{1,2\}$ of the monomial curves $C_1(11,15,18,35)$ and $C_2(20,25,39,41)$ which is given by Computer algebra system Macaulay created by D.Bayer and M.Stillman.

$M_1 = \{x_1^3-x_2x_3, x_2^5-x_1^2x_3x_4, x_3^4-x_1^2x_2x_4, x_4^2-x_1^2x_2^2x_3,$
$x_1x_3^3-x_2^2x_4, x_1x_2^4-x_3^2x_4\}$,
$M_2 = \{x_1^4-x_3x_4, x_2^4-x_1x_3x_4, x_3^4-x_1^2x_2^3x_4, x_4^3-x_1x_2x_3^2,$
$x_1^2x_3^3-x_2^3x_4^2, x_2x_3^3-x_1^3x_4^2\}$.

The ideals $P_t, t \in \{1,2\}$ generating by these sets belongs to the case of Theorem 4.1 for $(i,j,k,l)$=(2,3,4,1), $u$ = 1 and $(i,j,k,l)$=(3,4,2,1), $u$ = 0.

Hence ideals $P_t, t \in \{1,2\}$ is s.t.c.i.

$P_1 = Rad\left(g_1, x_1^3-x_2x_3, x_4^2-x_1^2x_2^2x_3\right)$, where
$g_1 = x_2^{18}-6x_1^2x_2^8x_3x_4+15x_1^4x_2^8x_3^2-20x_2^5x_3^5x_4^3+\sum_{h=4}^{6}$
$(-1)^{6+h}\binom{6}{h}x_3^{4h-9}\left(x_1^2x_2x_4\right)^{6-h}$ and
$P_2 = Rad\left(g_2, x_1^4-x_3x_4, x_2^4-x_1x_3x_4\right)$, where
$g_2 = x_3^{41}-16x_1^2x_2^3x_3^{37}x_4+120x_1^5x_2^2x_3^{34}x_4^3-560x_1^4x_2x_3^{321}x_4^6+$
$\sum_{h=4}^{16}(-1)^{16+h}\binom{16}{h}x_4^{3h-9}\left(x_1x_2x_3^2\right)^{16-h}$.

## References


[1] KUNZ, E.: *Einfuhrung in die kommutative Algebra und algebraische Geometrie.* Vieweg : Braunschweig, 1980.

[2] BRESINSKY, H.: Monomial Gorenstein Curves in $A^4$ as Set-theoretic Complete Intersections. *Manuscripta Math,* 27, 1979, 353-358.

[3] PATIL, D.: Certain Monomial Curves are Set-theoretic Complete Intersections. *Manuscripta Math.,* 68, 1990, 399-404.

[4] SOLCAN, S.: *Monomial Curves $C(p^2, p^2+p, p^2+p+1, (p+1)^2)$ as Set-theoretic Complete Intersections [in Slovak].* Proc. of intern. scientific conference: Mathematics in education, research and practice, Nitra, 2002, 94-97.

[5] SOLCAN, S.: Monomial Curves and Set Complete Intersections. *Slovak J. for Geometry and Graphics,* vol, 8, No. 15, 2011, 29-40.

[6] GASTINGER, W.: *Uber die Verschwindungsideale monomialer Kurven.* Dissertation zur Erlangung des Doktorgrades der Naturwissenschaften, Landshut, 1989.

[7] HOLESOVA, M: Asociated Prime Ideals of Monomial Curves with Five Generators as Set-theoretic Complete Intersections. *Studies of the University of Zilina, Mathematical Series,* vol. 25, 2011, 1-8.

[8] BRESINSKY, H.: Binomial Generating Sets for Monomial Curves, with Applications in $A^4$. Rend.Sem.Mat., 46, 1988, pp. 353-370, Univers. Politecn. Torino.

[9] TURCANOVA, M.: *Some Monomial Curves in $A^4$ as Set-theoretic Complete Intersections [in Slovak].* RNDr. Thesis, FMFI UK : Bratislava, 2000.

[10] HOLESOVA, M: Some Monomial Curves as Set-theoretic Complete Intersections. *Acta Math.,* 12, 2004, 33-39, Universitatis Ostraviensis : Ostrava.

[11] HOLESOVA, M: *Associated Ideals of Monomial Curves in $A^4$ [in Slovak].* PhD. thesis, FMFI UK : Bratislava, 2005.

[12] COX, D., LITTLE, J., O'SHEA, D.: Using Algebraic Geometry. *Graduate Texts in Mathematics*, vol. 185, Springer - Verlag, 1998.