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THE ACCELERATION WAVE IN A THIN TWO-MATERIAL AND THREE-SEGMENTAL ROD WITH SLOWLY CHANGING CROSS-SECTION MADE OF MURNAGHAN MATERIAL

The paper presents propagation of acceleration waves in a thin rod with slowly changing cross-section made of hyperelastic Murnaghan material. Analytical calculations were made based on the models obtained for the intensity of the transmitted and reflected wave. Numerical analysis of the velocity of wave propagation in the three-segmental rod made of two compressible elastic materials (steel and aluminum) has been made.

Keywords: *Murnaghan material, acceleration wave, compressible material, FEM.*

1. Introduction

A mathematical model of continuum is used for investigations in numerous fields of science, whereas the results obtained are used in practice in different aspects of technological solutions. However, finding a relatively simple constitutive relationship, which contains small number of constants and can be used in modelling of material properties in the full range of deformation, remains the subject of many studies and analyses.

Constitutive equations have the nature of phenomenological relations determined usually by means of experimental studies and represent a specific link that connects deformation of the material caused by the stress. These equations describe the relationships between deformations and stresses or between deformations and energy for hyperelastic materials are obtained based on the equations of mechanical energy balance. The model of material is adopted depending on the factors that are of essential importance to behaviour of the specific medium. The most frequently used constitutive relationship for compressible materials is Murnaghan elastic potential.

In terms of the theory of elasticity and in broadly understood mechanical problems, including continuum mechanics, elastic bodies are considered as material continuum with internal bonds and without them. For the elastic bodies without bonds, the properties of such a medium are given if the function W can be defined, which, for any deformation d of this medium, determines the corresponding elastic energy $W=W(d)$ accumulated in the unit of volume with respect to the reference configuration B_R.

Function W is typically defined as a function of deformation energy. For uniform isotropic elastic bodies, the constitutive equations can be written as

$$W = W(I_1, I_2, I_3) \tag{1}$$

where I_1, I_2, I_3 are invariants of the deformation tensor.

2. The acceleration wave in a thin segmental rod with slowly varying cross-section

The basic equations of propagation for the rod discussed originate from the study [1]. According to this study, in the case of propagation of the acceleration wave in a hyperelastic rod with variable cross-section (Fig. 1), the equation of transport for the wave intensity is a generalized Riccati equation. A supplementation of the above publication is the study [2] where authors demonstrated that the intensity of the acceleration wave that propagates in a rod with variable cross-section, with elastic potential Σ(p) meeting the condition $\partial^3 \sum / \partial p^3 \neq 0$, is represented by the Bernouli equation (p denotes displacement gradient).

We consider a homogeneous isotropic elastic material with elastic potential Σ and constant density ρ_R. An axially-symmetric, semi-infinite thin rod is assumed, with its cross-section varying slowly with the distance. A Cartesian system of material coordinates $\{X_\alpha\}$ in the reference configuration B_R and Cartesian system of spatial coordinates $\{x_i\}$ in the current configuration B are also assumed for $(i, \alpha = 1, 2, 3)$, which parameterize the same space and are mutually covered. The deformation of the rod is (see [1])

$$x = u(X, t) + X \quad \text{for} \quad x_1 = x \tag{2}$$

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where *u(X,t)* is a displacement of the points of continuum in the direction of the rod axis.

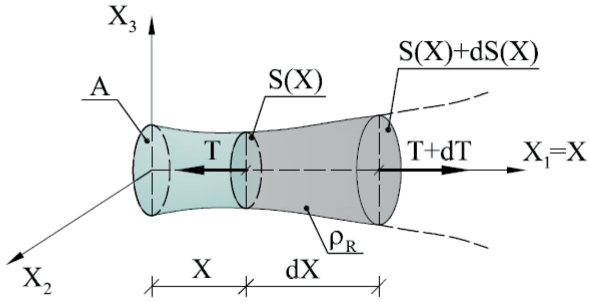


Fig. 1 Axially-symmetric rod with slowly varying cross-section

For the deformation (2), the equation of motion is (see [1])

$$\frac{\partial v}{\partial t} - c^2 \frac{\partial p}{\partial X} - \frac{T}{\rho_R} - \frac{d}{dX} \ln S(X) = 0 \qquad (3)$$

where $c = c(p) = \left(\frac{1}{\rho_R} \frac{\partial^2 \Sigma}{\partial p^2} \right)^{\frac{1}{2}}$ (4)

The above term Σ means elastic energy accumulated per unit of surface. For the case of the rod considered, the condition (3) represents basic equations of single-dimensional propagation of the acceleration wave.

2.1 Propagation of the acceleration wave

We assume that the propagation of disturbance occurs along the rod axis in the direction of the axis *X*. The area before the front remains undisturbed and is subjected to the deformation $p_o(X)$. Functions *p* and *v* are continuous, the discontinuity jump occurs for their first and second derivatives.

We assume an exponential change in the rod cross-section $S(X) = A_o e^{\gamma \cdot x}$ for $\sigma_1(X)$ and $S(X) = A_o e^{-\gamma \cdot x}$ for $\sigma_2(X)$. Therefore, the acceleration wave intensity σ*(X)* is (see [2] and [3])

$$\sigma_{1,2}(X) = \sigma(0) \frac{e^{\pm \frac{Q(X) \cdot X \cdot \gamma}{2}}}{1 \mp \sigma(0) \left[2\left(e^{\pm \frac{Q(X) \cdot X \cdot \gamma}{2}} \right) \frac{\beta(X)}{Q(X) \cdot \gamma} \right]} \qquad (5)$$

where $Q(X) = \left[1 - \frac{T_0}{2\rho_R^2 c_o^4} \left(\frac{\partial^3 \Sigma}{\partial p^3} \right) \right]$ (6)

σ*(0)* is an initial value of the acceleration wave intensity for *X=0*, whereas

$$\mu(X) = \frac{1}{2} \left[1 - \frac{T_0}{2\rho_R^2 c_o^4} \left(\frac{\partial^3 \Sigma}{\partial p^3} \right) \right] \left[\frac{d}{dX} \ln S(X) \right]$$

and $\beta(X) = -\frac{1}{4\rho_R c_o^4} \left(\frac{\partial^3 \Sigma}{\partial p^3} \right)$ (7)

The above term c_o represents the velocity of propagation of the front of acceleration wave [2].

2.2 Acceleration wave for Murnaghan material

We assume that the acceleration wave propagates in a compressible elastic material determined by Murnaghan potential [4]

$$W(I_1, I_2, I_3) = \rho_R \Sigma(I_1, I_2, I_3) = \frac{l + 2m}{24}(I_1 - 3)^3 + \\ + \frac{\lambda + 2\mu + 4m}{8}(I_1 - 3)^2 + \frac{8\mu + n}{8}(I_1 - 3) - \frac{m}{4} \\ (I_1 - 3)(I_2 - 3) - \frac{4\mu + n}{8}(I_2 - 3) + \frac{n}{8}(I_1 - 1), \qquad (8)$$

where λ and μ are Lame's constants, *l,m,n* - elastic constants of the second order. We assume these constants according to Table 1 (see [5]).

Elastic constants for steel and aluminum [5] Table 1

Material	λ [GPa]	μ [GPa]	*l* [GPa]	*m* [GPa]	*n* [GPa]
Steel	108.854	80.513	-452.087	-623.703	-694.311
Aluminum	55.898	27.066	-304.987	-393.247	-400.111

3. Calculations for two-material and three-segmental rod

For the analysis of propagation of the surface of discontinuities, modelled as a flat acceleration wave, we assumed a thin, axially-symmetric three-segmental rod with slowly varying cross-section. Each of the three segments, at the lengths of l_i, for *i=1,2,3* (Fig. 2), were made of homogeneous hyperelastic materials with isotropic properties. c_i and ρ_i denote velocity of propagation of the longitudinal waves and the density of the material. It was assumed that the cross section *S(X)* changes with the distance. We assume that, at the instant *t=0* for *X=0*, propagation of the surface of discontinuities occurs along the increasing values of the axis *X*, whereas the area before the wave front remains undisturbed $v(X,t)=0$. The functions of the velocity of propagation ν and gradient of displacement *p*, are according to [3].

3.1 Intensity of the acceleration wave for the cases of the rod analysed

It was assumed that σ(X) is the intensity of the wave at the point *X* of the rod studied. For γ>0 and γ<0, intensity of the acceleration wave is given by (5). For analytical calculations, we assume a static velocity of deformation of the order of 10 s^{-1}.

It was assumed that the shape of the rod is described, for the first segment and the half of the second segment by $S(X) = A_o e^{-\gamma \cdot x}$, and, from the half of the segment 2 and in the third segment, by the function $S(X) = A_o e^{\gamma \cdot x - 0.3}$.

Based on the conditions on the division plane, we assume that

$$\left.\frac{\sigma^{T2}(X^+)}{\sigma_1(X^-)}\right|_{X=l_1} = \frac{2\rho_{R1}c_1}{\rho_{R1}c_1 + \rho_{R2}c_2}, \quad \left.\frac{\sigma^{T3}(X^+)}{\sigma 2(X^-)}\right|_{X=l2} = \frac{2\rho_{R2}c2}{\rho_{R2}c_2 + \rho_{R3}c_3} \tag{9}$$

$$\left.\frac{\sigma^{R1}(X^-)}{\sigma_1(X^-)}\right|_{X=l_1} = \frac{\rho_{R1}c_1 - \rho_{R2}c_2}{\rho_{R1}c_1 + \rho_{R2}c_2}, \quad \left.\frac{\sigma^{R2}(X^-)}{\sigma 2(X^-)}\right|_{X=l2} = \frac{\rho_{R2}c_2 - \rho_{R3}c_3}{\rho_{R2}c_2 + \rho_{R3}c_3} \tag{10}$$

X^+ is a coordinate on the surface Λ before the front of the acceleration wave, whereas X^- is the coordinate on the same surface Λ behind the front of the acceleration wave.

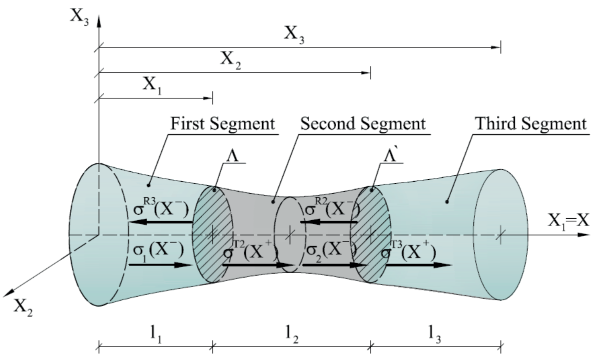
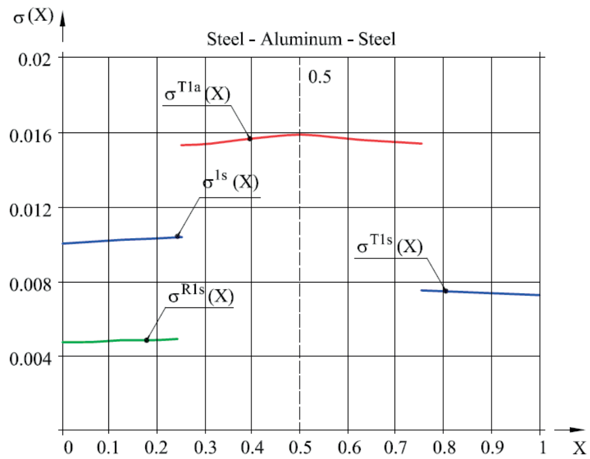
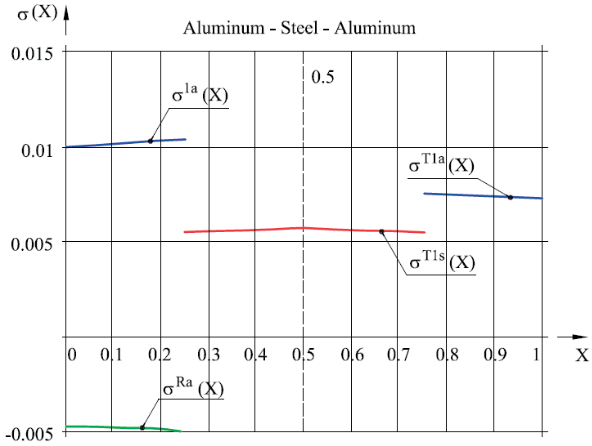


Fig. 2 Flat acceleration wave (falling σ(X), reflected $\sigma^R(X)$ and transmitted $\sigma^T(X)$) that propagates in two-material three-segmental rod with varying cross-section

We assumed two non-linear compressible elastic materials described with Murnaghan potential [6]. Cross-section of the rod changes exponentially, and this change is described by the function $S(X) = A_o e^{-\gamma \cdot x}$. It was assumed that in the second segment, the change in the cross-section of the rod occurs from decreasing into increasing (see Fig. 2)



a) Case one



b) Case two

Fig. 3 Intensities [m/s²] of the acceleration wave (falling σ(X), transmitted $\sigma_T(X)$ and reflected $\sigma_R(X)$) for three-segmental two-material rod with the length of $X \in \langle 0;1 \rangle$ m. Superscripts s and a correspond to steel and aluminum, respectively

The calculations for the three-segmental and two-material rod showed that the change in the cross-section of the rod causes the change in the intensity of the propagating acceleration wave. This is particularly noticeable in the second segment of the rod where the increasing and decreasing function is observed. Due to different density of the materials at the sections l_1, l_2, l_3, which are 0.25 m, 0.5 m and 0.25 m, respectively, the intensity jumps are observed in the surface of contact of individual segments for the propagating acceleration wave. The charts presented in Fig. 3a show that the intensity of the acceleration wave that goes through aluminum to the steel decreases. After transition from the steel to aluminum, it increases. This causes that the intensity of the acceleration wave that propagates in the hyperelastic material that moves from the material with lower density to the material with higher density is decreasing, whereas in the case of moving from the material with higher density towards the material with lower density, the intensity increases. The initial value of the intensity of the acceleration wave of 0.01 m/s^2 after moving through the first segment of the rod has at the beginning of the second segment the value higher in aluminum ~0.01526 m/s^2 and lower in steel ~0.0055 m/s^2. This leads to the conclusion that the intensity of the wave transmitted in aluminum increases by ca. 50% compared to the intensity of the wave propagating in steel and decreases in steel also by ca. 50% compared to the wave propagating in aluminum.

3.2 Numerical example

In order to carry out the numerical modelling of the course of wave (disturbance) propagation in an isotropic, two-material and three-segmental rod with slowly changing cross-section, we adopted two 2D objects which are the cross-section of the rod with the length of 1m (see Fig. 4). The initial and final diameters of the rod are 10 cm, whereas the middle diameter in 9.2 cm and, at the contact of segments (the first with the second and the second with the third), it is 9.6 cm. Discretization of the object discussed was carried out by means of 4 nodes. Due to the variety of rod diameters in the frontal and middle planes and at the surface of the contact of materials, generation of the grid forced, in several points, creation of 3 node objects. In order, for the rod analysed, we obtained 1000 elements 2D. The load of the objects analysed was represented by forced linear dislocation of the edges L11 towards the y axis. The value of the forced displacement amounted to 1cm for the time t=0.01 s. On the opposite (with respect to the load) L1 edge, the bonds were imposed to prevent from displacement towards y and z axes. These are bonds {B}, see Fig. 4.

The aim of the numerical calculations was to model the propagation of the wave (disturbance) for the three-segmental rod made of two compressible elastic materials (steel and aluminum). ADINA software was used to present the problems referring to analytical computations performed in this paper (see also [7]). ADINA is a program often used for stress and velocity analysis [8]. Calculations were made for the above boundary conditions and the assumed deformation. The declared load in the object occurred at the instant t=0.01 s, and then the effect was removed. The process of propagation of the disturbance in the model could be observed in 100 other steps. The dynamic implicit method was used. Therefore, we obtained information about generation of the velocity of disturbance propagation in two-material and three-segmental compressible rods with slowly varying cross-section.

For the given boundary conditions and external effects (displacements), Figs. 5 and 6 show the course of propagation of disturbance in two-material and three-segmental rod for selected time steps.

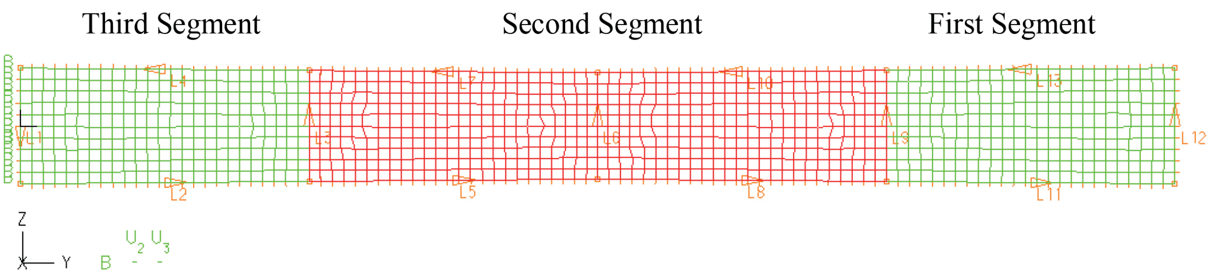


Fig. 4 2D model of a rod: grid, load and boundary conditions

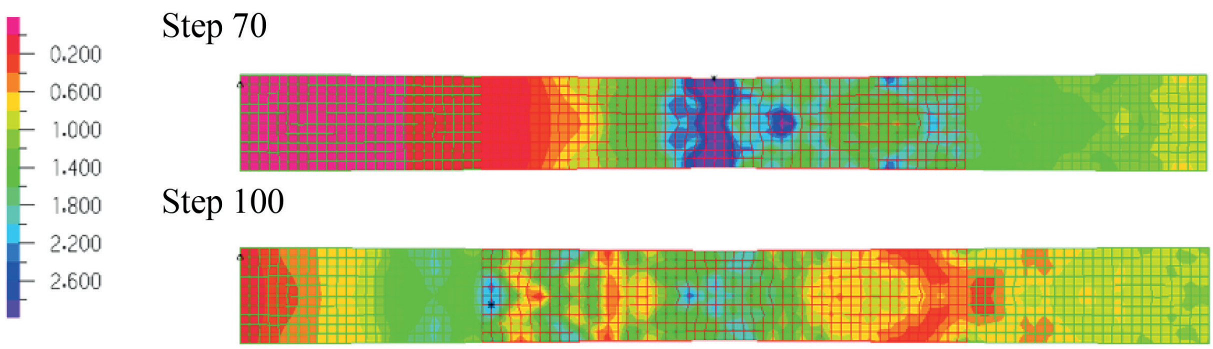


Fig. 5 The velocity of wave propagation [m/s] in steel (on the right), aluminum (in the middle) and steel (on the left), respectively

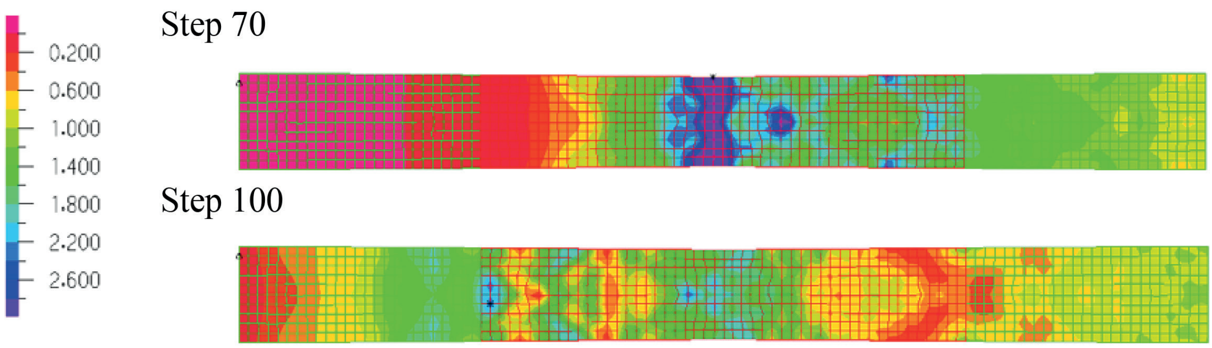


Fig. 6 The velocity of wave propagation [m/s] in aluminum (on the right), steel (in the middle) and aluminum (on the left), respectively

4. Conclusions

The comparative analysis carried out for the velocity of propagation of the wave (disturbance) in the rod with increasing and decreasing cross-section obtained using the adopted numerical model reveals that there are differences in the course of wave propagation in steel and aluminum. The results obtained confirm the results obtained through analytical investigations. It can be noticed that the wave velocity varies depending on the cross-section for the rod configurations analysed. The propagating wave, moving from the material with lower density (aluminum – 2700 kg/m^3) to the material with the greater density (steel – 7859 kg/m^3) decreases the velocity of propagation, see Fig. 5, step 100, and Fig. 6, step 70. At the moment of transition from the segment 1 to 2 a change in the velocity of the propagating transmitted wave can be observed (i.e. during transition from steel to aluminum, see Fig. 5 and from aluminum to steel, see Fig. 6). With transition from steel to aluminum, the velocity increases from ~1.4 m/s to ~2.6 m/s, whereas with transition from aluminum to steel, it decreases from ~1.625 m/s to ~0.875 m/s. Similar pattern is observed for the velocity of the transmitted wave for the transition from the segment 2 to segment 3. With transition from steel to aluminum, the velocity of the propagated wave increases (Fig. 6, step 100), whereas with transition from aluminum to steel, it decreases (Fig. 5, step 100). At the moment of transition from the material I to the material II and the material II to the material III, the wave propagating in the rod divides the energy into the transmitted and reflected waves. The reflected wave can be observed in both figures above (Fig. 5, step 70 and Fig. 6, step 70).

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