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STRESS SENSITIVITY ANALYSIS OF THE BEAM AND SHELL FINITE ELEMENTS

The stress sensitivity analysis in conjunction with finite elements method represents an important tool for the influence analysis of the structural parameters. This analysis plays a significant role in the decision process of the formulation of the structural optimizing or probability analysis. The goal of the paper is to present theoretic and numerical aspects of the beam and shell element stress sensitivity analysis with the respect to the cross section parameters (cross section area, thickness, etc.). The whole computational procedure was inbuilt into Matlab's software module MATFEM.

Keywords: stress sensitivity analysis, beam, thin shell element, Matlab

1. Introduction

Nowadays the sensitivity analysis is a significant tool helping to realize a structural parameters influence analysis. This analysis is usually very computer time consuming but the results are very innovative. This process is often applied to a structural analysis, i.e. in stress and strain analysis, modal and spectral or buckling analysis, stochastic analysis and so on [3, 6].

Application of the sensitivity analysis is not associated only with the structural optimizing but also with the analysis of the mechanical systems with uncertain parameters, mainly in the usage of so-called perturbation methods based on differentiation of the response with respect to the uncertain system parameters (stiffness, mass, damping, etc.). Implementation of this computational process into the finite element method characterized mainly the era of development of structural optimizing techniques in the eighties.

2. Stress sensitivity analysis for beam finite element

We will consider classic linear two-nodes beam element with a constant cross section (There is more information in [1, 5].).

Let's consider only a well-known linear distribution of the normal stress, i.e.

$$\sigma_x = \frac{N_x}{A} - \frac{M_{oz}}{J_z} \cdot y + \frac{M_{oy}}{J_y} \cdot z\,, \tag{1}$$

where N_x is the internal axial force, M_{oz} and M_{oy} are bending moments, A is the element cross-section area, J_z and J_y are moments of inertia [2, 5, 7, 8].

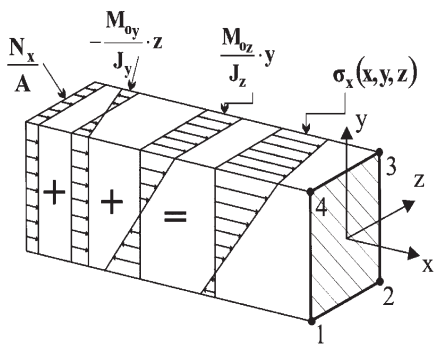

Fig. 1 Stress distribution in beam

If the bending moment is [1, 6]

$$M_o(x) = \left[\left(1 - \frac{x}{l}\right) \frac{x}{l}\right] \cdot \begin{bmatrix} M_{o1} \\ M_{o2} \end{bmatrix}, \tag{2}$$

than the normal stress function is following

$$\sigma_x(x,y,z) = \left\{0\,0\,0\,0\left[\frac{z}{J_y}\cdot\left(1-\frac{x}{l}\right)\right]\left[-\frac{y}{J_z}\cdot\left(1-\frac{x}{l}\right)\right]\cdot\right.$$
$$\left.\cdot\left(\frac{1}{A}\right)0\,0\,0\left(\frac{z}{J_y}\cdot\frac{x}{l}\right)\left(-\frac{y}{J_z}\cdot\frac{x}{l}\right)\right\}\cdot \tag{3}$$
$$\cdot\left[N_{x1}\; T_{y1}\; T_{z1}\; M_{k1}\; M_{y1}\; M_{z1} N_{x2}\; T_{y2}\; T_{z2}\; M_{k2}\; M_{y2}\; M_{z2}\right]^T \quad .$$

In agreement with points in Fig. 1 it is possible to write the relationship between normal stress in these marginal points of the cross-section and internal elements forces and moments, i.e.

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$$\begin{Bmatrix}\sigma_{x11}\\ \sigma_{x12}\\ \sigma_{x13}\\ \sigma_{x14}\\ \sigma_{x21}\\ \sigma_{x22}\\ \sigma_{x23}\\ \sigma_{x24}\end{Bmatrix} = \begin{Bmatrix} 0 & 0 & 0 & 0 & \frac{z_1}{J_y} & -\frac{y_1}{J_z} & \frac{1}{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{z_2}{J_y} & -\frac{y_2}{J_z} & \frac{1}{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{z_3}{J_y} & -\frac{y_3}{J_z} & \frac{1}{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{z_4}{J_y} & -\frac{y_4}{J_z} & \frac{1}{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{A} & 0 & 0 & 0 & \frac{z_1}{J_y} & -\frac{y_1}{J_z} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{A} & 0 & 0 & 0 & \frac{z_2}{J_y} & -\frac{y_2}{J_z} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{A} & 0 & 0 & 0 & \frac{z_3}{J_y} & -\frac{y_3}{J_z} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{A} & 0 & 0 & 0 & \frac{z_4}{J_y} & -\frac{y_4}{J_z} \end{Bmatrix} \cdot \begin{Bmatrix} N_{x1}\\ T_{y1}\\ T_{z1}\\ M_{k1}\\ M_{y1}\\ M_{z1}\\ N_{x2}\\ T_{y2}\\ T_{z2}\\ M_{k2}\\ M_{y2}\\ M_{z2}\end{Bmatrix} \quad (4)$$

or

$$\boldsymbol{\sigma}_L^j = \mathbf{C}_L^j \cdot \mathbf{f}_L^j\,, \quad (5)$$

The stress sensitivity centre is the derivative (5) with respect to a design variable X_i, i.e.

$$\frac{\partial \boldsymbol{\sigma}_L^j}{\partial X_i} = \frac{\partial \mathbf{C}_L^j}{\partial X_i} \cdot \mathbf{f}_L^j + \mathbf{C}_L^j \cdot \frac{\partial \mathbf{f}_L^j}{\partial X_i}\,. \quad (6)$$

Using the well-known finite element theory, the internal forces vector $\mathbf{f}_L^j$ in the local coordinate system is given by

$$\mathbf{f}_L^j = \mathbf{K}_L^j \cdot \mathbf{u}_L^j = \mathbf{K}_L^j \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j \cdot \mathbf{K}_G^{-1} \cdot \mathbf{f}_G\,, \quad (7)$$

where $\mathbf{K}_G$ is the global stiffness matrix (in the global coordinate system), $\mathbf{f}_G$ is the external nodal forces vector (in the global coordinate system), $\mathbf{T}_{LG}^j$ is a transformation matrix between the local and global coordinate systems, $\mathbf{T}_{01}^j$ is a Boolean matrix, i.e. the localization matrix determining the element position in the global stiffness matrix, it means

$$\mathbf{u}_G^j = \mathbf{T}_{01}^j \cdot \mathbf{u}_G\,. \quad (8)$$

Let's now realize the derivation of the internal nodal forces vector (7) with respect to X_i:

$$\frac{\partial \mathbf{f}_L^j}{\partial X_i} = \frac{\partial \mathbf{K}_L^j}{\partial X_i} \cdot \mathbf{u}_L^j + \mathbf{K}_L^j \cdot \frac{\partial \mathbf{u}_L^j}{\partial X_i} = \frac{\partial \mathbf{K}_L^j}{\partial X_i} \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j \cdot \mathbf{K}_G^{-1} \cdot \mathbf{f}_G + \mathbf{K}_L^j \cdot \frac{\partial \mathbf{u}_L^j}{\partial X_i}\,. \quad (9)$$

Applying the derivation on the well-known "FEA" equation $\mathbf{K}_G \cdot \mathbf{u}_G = \mathbf{f}_G$ we can write

$$\frac{\partial \mathbf{K}_G}{\partial X_i} \cdot \mathbf{u}_G + \mathbf{K}_G \cdot \frac{\partial \mathbf{u}_G}{\partial X_i} = \frac{\partial \mathbf{f}_G}{\partial X_i}\,. \quad (10)$$

where

$$\frac{\partial \mathbf{K}_G}{\partial X_i} = \sum_{j=1}^{ni} \mathbf{T}_{01}^{j\,T} \cdot \mathbf{T}_{LG}^{j\,T} \cdot \frac{\partial \mathbf{K}_L^j}{\partial X_i} \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j\,, \quad (11)$$

and n_i is a number of all the elements containing X_i. Further, the gradient of a global vector of the nodal displacements can be following

$$\frac{\partial \mathbf{u}_G}{\partial X_i} = \mathbf{K}_G^{-1} \cdot \left(\frac{\partial \mathbf{f}_G}{\partial X_i} - \frac{\partial \mathbf{K}_G}{\partial X_i} \cdot \mathbf{u}_G\right) = \mathbf{K}_G^{-1} \cdot \left[\frac{\partial \mathbf{f}_G}{\partial X_i} - \right. \left. - \sum_{j=1}^{ni}\left(\mathbf{T}_{01}^{j\,T} \cdot \mathbf{T}_{LG}^{j\,T} \cdot \frac{\partial \mathbf{K}_L^j}{\partial X_i} \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j\right) \cdot \mathbf{u}_G\right] = \mathbf{K}_G^{-1} \cdot \left[\frac{\partial \mathbf{f}_G}{\partial X_i} - \sum_{j=1}^{ni}\left(\mathbf{T}_{01}^{j\,T} \cdot \mathbf{T}_{LG}^{j\,T} \cdot \frac{\partial \mathbf{K}_L^j}{\partial X_i} \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j\right) \cdot \mathbf{K}_G^{-1} \cdot \mathbf{u}_G\right]. \quad (12)$$

Relationship between $\mathbf{u}_L^j$ and $\mathbf{u}_G$ will be given by

$$\frac{\partial \mathbf{u}_L^j}{\partial X_i} = \mathbf{T}_{LG}^j \cdot \frac{\partial \mathbf{u}_G^j}{\partial X_i} = \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j \cdot \frac{\partial \mathbf{u}_G}{\partial X_i} = \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j \cdot \mathbf{K}_G^{-1} \cdot \left[\frac{\partial \mathbf{f}_G}{\partial X_i} - \sum_{j=1}^{ni}\left(\mathbf{T}_{01}^{j\,T} \cdot \mathbf{T}_{LG}^{j\,T} \cdot \frac{\partial \mathbf{K}_L^j}{\partial X_i} \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j\right) \cdot \mathbf{K}_G^{-1} \cdot \mathbf{f}_G\right]. \quad (13)$$

Substituting (13) into (9), we can obtain the derivation of the $\mathbf{f}_L^j$ with respect to X_i as follows

$$\frac{\partial \mathbf{f}_L^j}{\partial X_i} = \frac{\partial \mathbf{K}_L^j}{\partial X_i} \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j \cdot \mathbf{K}_G^{-1} \cdot \mathbf{f}_G + \mathbf{K}_L^j \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j \cdot \mathbf{K}_G^{-1} \cdot \left[\frac{\partial \mathbf{f}_G}{\partial X_i} - \sum_{j=1}^{ni}\left(\mathbf{T}_{01}^{j\,T} \cdot \mathbf{T}_{LG}^{j\,T} \cdot \frac{\partial \mathbf{K}_L^j}{\partial X_i} \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j\right) \cdot \mathbf{K}_G^{-1} \cdot \mathbf{f}_G\right]. \quad (14)$$

Finally, after the substituting (14) into (6) we give the gradient of the j-th element stress vector

$$\frac{\partial \boldsymbol{\sigma}_L^j}{\partial X_i} = \frac{\partial \mathbf{C}_L^j}{\partial X_i} \cdot \mathbf{K}_L^j \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j \cdot \mathbf{K}_G^{-1} \cdot \mathbf{f}_G + \mathbf{C}_L^j \cdot \left\{\frac{\partial \mathbf{K}_L^j}{\partial X_i} \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j \cdot \mathbf{K}_G^{-1} \cdot \mathbf{f}_G + \mathbf{K}_L^j \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j \cdot \mathbf{K}_G^{-1} \cdot \left[\frac{\partial \mathbf{f}_G}{\partial X_i} - \sum_{j=1}^{ni}\left(\mathbf{T}_{01}^{j\,T} \cdot \mathbf{T}_{LG}^{j\,T} \cdot \frac{\partial \mathbf{K}_L^j}{\partial X_i} \cdot \mathbf{T}_{LG}^j \cdot \mathbf{T}_{01}^j\right) \cdot \mathbf{K}_G^{-1} \cdot \mathbf{f}_G\right]\right\}. \quad (15)$$

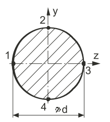
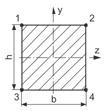
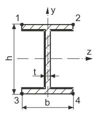
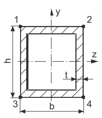
It should be noted that the derivation of the matrix $\boldsymbol{C}_j$ with respect to X_i, depends on the used cross-section. To realize the derivation $\frac{\partial \boldsymbol{\sigma}_L^j}{\partial X_i}$ means to find the following derivations $\frac{\partial \mathbf{K}_L^j}{\partial X_i}$, $\frac{\partial \mathbf{C}_L^j}{\partial X_i}$ and $\frac{\partial \mathbf{f}_G}{\partial X_i}$. The last derivation usually equals to zero or is not significant for the sensitivity analysis.

Let the cross-section area be variable X_i then other cross-section characteristics will be expressed as follows

$$J_y = a \cdot X_i^p,\; J_z = b \cdot X_i^q,\; J_y = a \cdot X_i^r\,, \quad (16)$$

where parameters a, b, c and exponents p, q, r will be obtained exactly (a simple cross-section) or numerically (a more complicated cross section), using the least squares method. The values of these parameters are presented in Tab. 1.

Parameters *a, b, c* and exponents *p, q, r* for chosen cross sections Tab. 1

Cross-section	$J_y = a \cdot X_i^p$	$J_z = b \cdot X_i^q$	$J_y = a \cdot X_i^r$
Circle 	$a = 0.0796$	$b = 0.0796$	$c = 0.1592$
	$p = 2$	$q = 2$	$r = 2$
Square 	$a = 0.0833$	$b = 0.0833$	$c = 0.14$
	$p = 2$	$q = 2$	$r = 2$
I 	$a = 1.4389$	$b = 0.7947$	$c = 0.0094$
	$p = 2.0401$	$q = 1.7588$	$r = 2.0276$
Box 	$a = 11.8364$	$b = 13.8364$	$c = 2.7273$
	$p = 1.6047$	$q = 1.6217$	$r = 1.3592$

The derivation $\dfrac{\partial \mathbf{K}_L^j}{\partial X_i}$ can be given by

$$\frac{\partial \mathbf{K}_L^j}{\partial X_i} = \frac{\delta_{ij}}{X_i} \cdot \left[\mathbf{K}_1^j(X_i) + p \cdot \mathbf{K}_2^j\left(a \cdot X_i^p\right) + \right. \\ \left. + q \cdot \mathbf{K}_3^j\left(b \cdot X_i^q\right) + r \cdot \mathbf{K}_4^j\left(c \cdot X_i^r\right) \right], \quad (17)$$

where δ_{ij} is Kronecker delta and matrices $\mathbf{K}^j_{1,2,3,4}$ are sub-matrices of the stiffness matrix corresponding axial, bending (about axes y and z) and torsion stiffness of the used cross section (More information about stiffness parameters is in [1, 5, 6]).

Let's now focus on the derivation $\dfrac{\partial \mathbf{C}_L^j}{\partial X_i}$. The transformations matrix $\mathbf{C}_L^j$ depends on coordinates of the marginal points 1, 2, 3, 4, which have to be expressed by the variable X_i. Considering these conditions we can express matrix $\mathbf{C}_L^j$ for circular section as follows

$$\mathbf{C}_L^i = \left\{ \begin{matrix} 0 & 0 & 0 & 0 & -\frac{4\cdot\sqrt{\pi}}{\sqrt{X_i^3}} & 0 & \frac{1}{X_i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{4\cdot\sqrt{\pi}}{\sqrt{X_i^3}} & \frac{1}{X_i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4\cdot\sqrt{\pi}}{\sqrt{X_i^3}} & 0 & \frac{1}{X_i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{4\cdot\sqrt{\pi}}{\sqrt{X_i^3}} & \frac{1}{X_i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{X_i} & 0 & 0 & 0 & -\frac{4\cdot\sqrt{\pi}}{\sqrt{X_i^3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{X_i} & 0 & 0 & 0 & 0 & \frac{4\cdot\sqrt{\pi}}{\sqrt{X_i^3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{X_i} & 0 & 0 & 0 & \frac{4\cdot\sqrt{\pi}}{\sqrt{X_i^3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{X_i} & 0 & 0 & 0 & 0 & -\frac{4\cdot\sqrt{\pi}}{\sqrt{X_i^3}} \end{matrix} \right\} \quad (18)$$

and the analyzed derivation of this matrix gets the following form

$$\frac{\partial \mathbf{C}_L^i}{\partial X_i} = \left\{ \begin{matrix} 0 & 0 & 0 & 0 & \frac{6\cdot\sqrt{\pi}}{\sqrt{X_i^5}} & 0 & -\frac{1}{X_i^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{6\cdot\sqrt{\pi}}{\sqrt{X_i^5}} & -\frac{1}{X_i^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{6\cdot\sqrt{\pi}}{\sqrt{X_i^5}} & 0 & -\frac{1}{X_i^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{6\cdot\sqrt{\pi}}{\sqrt{X_i^5}} & -\frac{1}{X_i^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{X_i^2} & 0 & 0 & 0 & \frac{6\cdot\sqrt{\pi}}{\sqrt{X_i^5}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{X_i^2} & 0 & 0 & 0 & 0 & -\frac{6\cdot\sqrt{\pi}}{\sqrt{X_i^5}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{X_i^2} & 0 & 0 & 0 & -\frac{6\cdot\sqrt{\pi}}{\sqrt{X_i^5}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{X_i^2} & 0 & 0 & 0 & 0 & \frac{6\cdot\sqrt{\pi}}{\sqrt{X_i^5}} \end{matrix} \right\}. \quad (19)$$

It is possible to get exactly the previous matrices but, for example, $\mathbf{C}_L^j$ of the I-section has to be analyzed numerically [6].

Example 1

Let us consider the structural sensitivity analysis of normal stresses of the beam element from Fig. 2. Given: $E = 2.1e5$ MPa $F_1 = 800$ N, $F_2 = 6000$ N, $A_1 = 200$ mm^2, $A_2 = 35$ mm^2, $L_1 = 1$ m, $L_2 = 0.5$ m.

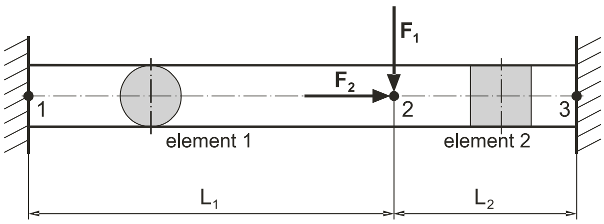


Fig. 2 Simple beam structure with 2 cross section

Applying previous relationships we can obtain the following derivations of matrices $\mathbf{C}_L^1$ and $\mathbf{C}_L^2$

$$\frac{\partial \mathbf{C}_L^1}{\partial X_1} = \begin{Bmatrix} 0 & 0 & 0 & -2.5e-5 & 0 & 0 \\ 0 & 0 & -1.8e-5 & -2.5e-5 & 0 & 0 \\ 0 & 0 & 0 & -2.5e-5 & 0 & 0 \\ 0 & 0 & -1.8e-5 & -2.5e-5 & 0 & 0 \\ 0 & 0 & 0 & -2.5e-5 & 0 & 0 \\ 0 & 0 & 0 & -2.5e-5 & 0 & -1.8e-5 \\ 0 & 0 & 0 & -2.5e-5 & 0 & 0 \\ 0 & 0 & 0 & -2.5e-5 & 0 & -1.8e-5 \end{Bmatrix},$$

$$\frac{\partial \mathbf{C}_L^2}{\partial X_2} = \begin{Bmatrix} 0 & 0 & -3.9e-6 & -8.16e-6 & 0 & 0 \\ 0 & 0 & -3.9e-6 & -8.16e-6 & 0 & 0 \\ 0 & 0 & -3.9e-6 & -8.16e-6 & 0 & 0 \\ 0 & 0 & -3.9e-6 & -8.16e-6 & 0 & 0 \\ 0 & 0 & 0 & -8.16e-6 & 0 & -3.9e-6 \\ 0 & 0 & 0 & -8.16e-6 & 0 & -3.9e-6 \\ 0 & 0 & 0 & -8.16e-6 & 0 & -3.9e-6 \\ 0 & 0 & 0 & -8.16e-6 & 0 & -3.9e-6 \end{Bmatrix}$$

and finally stress gradients in marginal points of the used cross-sections are the following

$$\frac{\partial \sigma_L^{(1)}}{\partial A_1} = \begin{Bmatrix} -0,0222 \\ -0,8077 \\ -0,0222 \\ 0,7632 \\ -0,0222 \\ -1,1897 \\ -0,0222 \\ 1,1452 \end{Bmatrix} \text{ and } \frac{\partial \sigma_L^{(1)}}{\partial A_1} = \begin{Bmatrix} -0,0222 \\ -0,8077 \\ -0,0222 \\ 0,7632 \\ -0,0222 \\ -1,1897 \\ -0,0222 \\ 1,1452 \end{Bmatrix} \text{ [MPa/mm}^2\text{]}.$$

The graphic presentation of the stress gradients is in Fig. 3. The presented stress gradient analysis was confronted with a "classical" numerical computational approach ($\Delta\sigma j / \Delta X_i$) and it's possible to observe the absolute consensus.

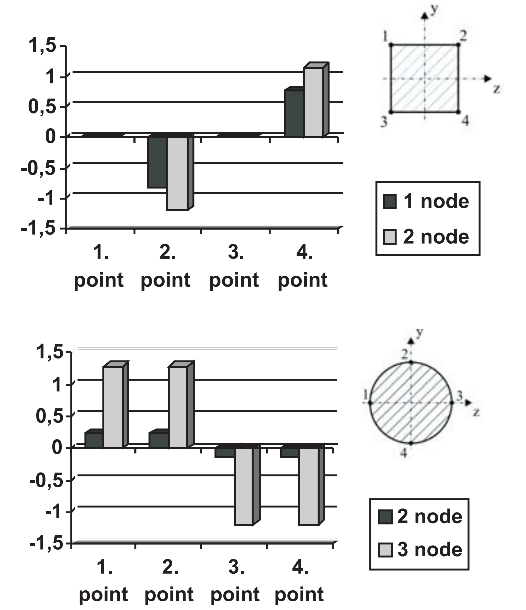


Fig. 3 Values of the stress gradient in the first and second element

3. Stress sensitivity analysis for a thin shell finite element

The finite element modeling of box, shell or thin-walled structures are usually realized using thin shell finite elements (Kirchhoff's or Mindlin's formulation) [1, 4, 9]. The stiffness parameters depend on material constants and element geometry, mainly on its thickness. Therefore, the thickness t_i will be the variable in the following theoretical and numerical stress sensitivity analysis of the shell finite element; the fundamental information about this analysis can be found in [1, 4, 9].

At first we have to prepare the stress calculation process. This process is based on the expression of the j-th element membrane forces and bending moments (without shear forces) [4, 6], i.e.

$$\begin{aligned} \left[F_{xx} \;\; F_{yy} \;\; F_{xy}\right]_j^T &= \mathbf{F}_m^j = \int_S \mathbf{E}_m^j \cdot \varepsilon_m^j dS_j = \\ &= \mathbf{E}_m^j \cdot \int_S \mathbf{B}_m^j dS_j \cdot \mathbf{u}_L^j = t_j \cdot \mathbf{D}_j \cdot \mathbf{I}_m^j \cdot \mathbf{u}_L^j \end{aligned} \tag{20}$$

and

$$\begin{aligned} \left[M_{xx} \;\; M_{yy} \;\; M_{xy}\right]_j^T &= \mathbf{M}_b^j = \int_S \mathbf{E}_b^j \cdot \varepsilon_b^j dS_j = \\ &= \mathbf{E}_b^j \cdot \int_S \mathbf{B}_b^j dS_j \cdot \mathbf{u}_L^j = \frac{t_j^3}{12} \cdot \mathbf{D}_j \cdot \mathbf{I}_b^j \cdot \mathbf{u}_L^j . \end{aligned} \tag{21}$$

The auxiliary matrices $\mathbf{I}_m$ and $\mathbf{I}_b$ can be calculated only using the numerical approach. Further details about $\mathbf{E}_m$, $\mathbf{E}_b$, $\mathbf{D}$, $\mathbf{B}_m$, $\mathbf{B}_b$, $\mathbf{u}_{el}$ and t are presented in [6]. The extreme stress values can be expected at the top or at the bottom surface. Generally, it means or in compliance with the previous beam element (eq. 5)

$$\begin{bmatrix} \sigma_{mb} \mid_{top} \\ \sigma_{mb} \mid_{bot} \end{bmatrix}^j = \begin{Bmatrix} \sigma_{xx,top} \\ \sigma_{yy,top} \\ \sigma_{xy,top} \\ \sigma_{xx,bot} \\ \sigma_{yy,bot} \\ \sigma_{xy,bot} \end{Bmatrix}^j = \begin{bmatrix} 1/t_j & 0 & 0 & 6/t_j^2 & 0 & 0 \\ 0 & 1/t_j & 0 & 0 & 6/t_j^2 & 0 \\ 0 & 0 & 1/t_j & 0 & 0 & 6/t_j^2 \\ 1/t_j & 0 & 0 & 6/t_j^2 & 0 & 0 \\ 0 & 1/t_j & 0 & 0 & 6/t_j^2 & 0 \\ 0 & 0 & 1/t_j & 0 & 0 & 6/t_j^2 \end{bmatrix} \cdot \begin{Bmatrix} F_{xx} \\ F_{yy} \\ F_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}_{t,top} \\ \mathbf{A}_{t,bot} \end{bmatrix}^j \cdot \begin{Bmatrix} \mathbf{F}_m \\ \mathbf{M}_b \end{Bmatrix}^j , \tag{22}$$

$$\boldsymbol{\sigma}^{j}_{mb\_L} = \mathbf{C}^{j}_{L} \cdot \mathbf{f}^{j}_{L} . \tag{23}$$

Let's build new material and auxiliary matrices

$$\mathbf{E}_{mb} = \begin{bmatrix} t_j \cdot \mathbf{I}_3 & 0_3 \\ 0_3 & \dfrac{t_j^3}{12} \cdot \mathbf{I}_3 \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{D} \\ \mathbf{D} \end{Bmatrix}_j = \mathbf{D}_t \cdot \mathbf{D}_{mb}, \; \mathbf{I}_{mb} = \begin{Bmatrix} \mathbf{I}^j_m \\ \mathbf{I}^j_b \end{Bmatrix}, \tag{24}$$

where the matrix $\mathbf{I}_3$ is the classical unit matrix. Then (22) can be written as follows

$$\boldsymbol{\sigma}_{j\_mb}\big|_{top} = \mathbf{A}_{t,top} \cdot \mathbf{E}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}^j_L = \mathbf{A}_{t,top} \cdot \mathbf{D}_t \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}^j_L , \tag{25a}$$

$$\boldsymbol{\sigma}_{j\_mb}\big|_{bot} = \mathbf{A}_{t,bot} \cdot \mathbf{E}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}^j_L = \mathbf{A}_{t,bot} \cdot \mathbf{D}_t \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}^j_L . \tag{25b}$$

Generally, the top or bottom von Mises stresses may be calculated from relations

$$\sigma^2_{j\_ekv}\big|_{top} = \boldsymbol{\sigma}^T_{j\_mb}\big|_{top} \cdot \mathbf{T}_{mb} \cdot \boldsymbol{\sigma}_{j\_mb}\big|_{top}$$

or

$$\sigma^2_{j\_ekv}\big|_{bot} = \boldsymbol{\sigma}^T_{j\_mb}\big|_{bot} \cdot \mathbf{T}_{mb} \cdot \boldsymbol{\sigma}_{j\_mb}\big|_{bot} \tag{26}$$

where

$$T_{mb} = \begin{bmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \tag{27}$$

Using (25) and (27) in (26) we obtain

$$\begin{aligned} \boldsymbol{\sigma}^2_{j\_ekv}\big|_{top} &= \boldsymbol{\sigma}^T_{j\_mb}\big|_{top} \cdot \mathbf{T}_{mb} \cdot \boldsymbol{\sigma}_{j\_mb}\big|_{top} = \\ &= \mathbf{u}^{jT}_L \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{D}^T_t \cdot \mathbf{A}^T_{t,top} \cdot \mathbf{T}_{mb} \cdot \mathbf{A}_{t,top} \cdot \mathbf{D}_t \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}^j_L = \\ &= \mathbf{u}^{jT}_L \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}^j_L \end{aligned} \tag{28a}$$

and

$$\begin{aligned} \sigma^2_{j\_ekv}\big|_{bot} &= \boldsymbol{\sigma}^T_{j\_mb}\big|_{bot} \cdot \mathbf{T}_{mb} \cdot \boldsymbol{\sigma}_{j\_mb}\big|_{bot} = \\ &= \mathbf{u}^{jT}_L \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{D}^T_t \cdot \mathbf{A}^T_{t,bot} \cdot \mathbf{T}_{mb} \cdot \mathbf{A}_{t,bot} \cdot \mathbf{D}_t \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}^j_L = \\ &= \mathbf{u}^{jT}_L \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}^j_L \end{aligned} \tag{28b}$$

where

$$T_{t,top} = \begin{bmatrix} 1 & -0.5 & 0 & 0.5 \cdot t_j & -0.25 \cdot t_j & 0 \\ -0.5 & 1 & 0 & -0.25 \cdot t_j & 0.5 \cdot t_j & 0 \\ 0 & 0 & 3 & 0 & 0 & 1.5 \cdot t_j \\ 0.5 \cdot t_j & -0.25 \cdot t_j & 0 & 0.25 \cdot t_j^2 & -0.125 \cdot t_j^2 & 0 \\ -0.25 \cdot t_j & 0.5 \cdot t_j & 0 & -0.125 \cdot t_j^2 & 0.25 \cdot t^2 & 0 \\ 0 & 0 & 1.5 \cdot t_j & 0 & 0 & 0.75 \cdot t_j^2 \end{bmatrix} \tag{29a}$$

and

$$T_{t,bot} = \begin{bmatrix} 1 & -0.5 & 0 & -0.5 \cdot t_j & 0.25 \cdot t_j & 0 \\ -0.5 & 1 & 0 & 0.25 \cdot t_j & -0.5 \cdot t_j & 0 \\ 0 & 0 & 3 & 0 & 0 & -1.5 \cdot t_j \\ -0.5 \cdot t_j & 0.25 \cdot t_j & 0 & 0.25 \cdot t_j^2 & -0.125 \cdot t_j^2 & 0 \\ 0.25 \cdot t_j & -0.5 \cdot t_j & 0 & -0.125 \cdot t_j^2 & 0.25 \cdot t_j^2 & 0 \\ 0 & 0 & -1.5 \cdot t_j & 0 & 0 & 0.75 \cdot t_j^2 \end{bmatrix} \tag{29b}$$

Assuming a relation between the local element displacements $\mathbf{u}^j_L$ and the global displacement vector $\mathbf{u}_G$

$$\mathbf{u}^j_L = \mathbf{T}_{LG} \cdot \mathbf{T}_{01} \cdot \mathbf{u}_G , \tag{30}$$

then (28a,b) may be rewritten as

$$\begin{aligned} \sigma^2_{j\_ekv}\big|_{top} &= \mathbf{u}^T_G \cdot \mathbf{T}^T_{01} \cdot \mathbf{T}^T_{LG} \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{T}_{t,top} \cdot \mathbf{D}_{mb} \cdot \\ &\cdot \mathbf{I}_{mb} \cdot \mathbf{T}_{LG} \cdot \mathbf{T}_{01} \cdot \mathbf{u}_G, \end{aligned} \tag{31a}$$

and

$$\begin{aligned} \sigma^2_{j\_ekv}\big|_{bot} &= \mathbf{u}^T_G \cdot \mathbf{T}^T_{01} \cdot \mathbf{T}^T_{LG} \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{T}_{t,bot} \cdot \mathbf{D}_{mb} \cdot \\ &\cdot \mathbf{I}_{mb} \cdot \mathbf{T}_{LG} \cdot \mathbf{T}_{01} \cdot \mathbf{u}_G \end{aligned} \tag{31b}$$

where $\mathbf{T}_{LG}$ is a classical transformation matrix between the local and the global coordinate systems, $\mathbf{T}_{01}$ is again a Boolean matrix, i.e. the localization matrix determining the element position in the global stiffness matrix.

The stress sensitivity analysis means the finding of von Mises stress derivative with respect to a chosen structural parameter, in our case the element thickness *t*. Let's analyze the differentiation of von Mises stress of *j*-th element with respect to the *i*-th element thickness t_i. Applying (31a, b) we can obtain

$$\begin{aligned} \frac{\partial \sigma^2_{j\_ekv}\big|_{top}}{\partial t_i} &= \frac{\partial \mathbf{u}^T}{\partial t_i} \cdot \mathbf{T}^T_{j\_01} \cdot \mathbf{T}^T_{j\_LG} \cdot \mathbf{I}^T_{j\_mb} \cdot \mathbf{D}^T_{j\_mb} \cdot \mathbf{T}_{j\_t,top} \cdot \\ &\cdot \mathbf{D}_{j\_mb} \cdot \mathbf{I}_{j\_mb} \cdot \mathbf{T}_{vLG} \cdot \mathbf{T}_{j\_01} \cdot \mathbf{u} + \mathbf{u}^T \cdot \mathbf{T}^T_{j\_01} \cdot \mathbf{T}^T_{j\_LG} \cdot \mathbf{I}^T_{j\_mb} \cdot \\ &\cdot \mathbf{D}^T_{j\_mb} \cdot \delta_{ij} \cdot \frac{\partial \mathbf{T}_{j\_t,top}}{\partial t_i} \cdot D_{j\_mb} \cdot \mathbf{I}_{j\_mb} \cdot \mathbf{T}_{j\_LG} \cdot \mathbf{T}_{j\_01} \cdot \mathbf{u} + \\ &+ \mathbf{u}^T \cdot \mathbf{T}^T_{j\_01} \cdot \mathbf{T}^T_{j\_LG} \cdot \mathbf{I}^T_{j\_mb} \cdot \mathbf{D}^T_{j\_mb} \cdot \mathbf{T}_{j\_t,top} \cdot \mathbf{D}_{j\_mb} \cdot \mathbf{I}_{j\_mb} \cdot \\ &\cdot \mathbf{T}_{j\_LG} \cdot \mathbf{T}_{j\_01} \cdot \frac{\partial \mathbf{u}}{\partial t_i}, \end{aligned} \tag{32a}$$

$$\frac{\partial \sigma^2_{j\_ekv}\big|_{bot}}{\partial t_i} = \frac{\partial \mathbf{u}^T}{\partial t_i} \cdot \mathbf{T}^T_{j\_01} \cdot \mathbf{T}^T_{j\_LG} \cdot \mathbf{I}^T_{j\_mb} \cdot \mathbf{D}^T_{j\_mb} \cdot \mathbf{T}_{j\_t,bot} \cdot$$
$$\cdot \mathbf{D}_{j\_mb} \cdot \mathbf{I}_{j\_mb} \cdot \mathbf{T}_{vLG} \cdot \mathbf{T}_{j\_01} \cdot \mathbf{u} + \mathbf{u}^T \cdot \mathbf{T}^T_{j\_01} \cdot \mathbf{T}^T_{j\_LG} \cdot \mathbf{I}^T_{j\_mb} \cdot$$
$$\cdot \mathbf{D}^T_{j\_mb} \cdot \delta_{ij} \cdot \frac{\partial \mathbf{T}_{j\_t,bot}}{\partial t_i} \cdot D_{j\_mb} \cdot \mathbf{I}_{j\_mb} \cdot \mathbf{T}_{j\_LG} \cdot \mathbf{T}_{j\_01} \cdot \mathbf{u} + \quad (32b)$$
$$+ \mathbf{u}^T \cdot \mathbf{T}^T_{j\_01} \cdot \mathbf{T}^T_{j\_LG} \cdot \mathbf{I}^T_{j\_mb} \cdot \mathbf{D}^T_{j\_mb} \cdot \mathbf{T}_{j\_t,bot} \cdot \mathbf{D}_{j\_mb} \cdot \mathbf{I}_{j\_mb} \cdot$$
$$\cdot \mathbf{T}_{j\_LG} \cdot \mathbf{T}_{j\_01} \cdot \frac{\partial \mathbf{u}}{\partial t_i},$$

where

$$\frac{\partial \mathbf{T}_{j\_t,top}}{\partial t_i} = \delta_{ij} \cdot \begin{bmatrix} 0 & 0 & 0 & 0.5 & -0.25 & 0 \\ 0 & 0 & 0 & -0.25 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.5 \\ 0.5 & -0.25 & 0 & 0.5 \cdot t_j & -0.25 \cdot t_j & 0 \\ -0.25 & 0.5 & 0 & -0.25 \cdot t_j & 0.5 \cdot t_j & 0 \\ 0 & 0 & 1,5 & 0 & 0 & 1.5 \cdot t_j \end{bmatrix} \quad (33a)$$

and

$$\frac{\partial \mathbf{T}_{j\_t,bot}}{\partial t_i} = \delta_{ij} \cdot \begin{bmatrix} 0 & 0 & 0 & -0,5 & 0,25 & 0 \\ 0 & 0 & 0 & 0,25 & -0,5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1,5 \\ -0,5 & 0,25 & 0 & 0,5 \cdot t_j & -0,25 \cdot t_j & 0 \\ 0,25 & -0,5 & 0 & -0,25 \cdot t_j & 0,5 \cdot t_j & 0 \\ 0 & 0 & -1,5 & 0 & 0 & 1,5 \cdot t_j \end{bmatrix} \quad (33a)$$

The derivative u with respect to ti may be expressed as

$$\frac{\partial \mathbf{u}_G}{\partial t_i} = \mathbf{K}_G^{-1} \cdot \left(\frac{\partial \mathbf{f}}{\partial t_i} - \frac{\partial \mathbf{K}_G}{\partial t_i} \cdot \mathbf{u}_G \right) \quad (34)$$

or, in more detail,

$$\frac{\partial \mathbf{u}_G}{\partial t_i} = \mathbf{K}_G^{-1} \cdot \left[\frac{\partial \mathbf{f}}{\partial t_i} - \left(\sum_{j=1}^{n} \mathbf{T}^T_{j\_01} \cdot \mathbf{T}^T_{j\_LG} \cdot \frac{\partial (\mathbf{K}_{j\_m} + \mathbf{K}_{j\_b} + \mathbf{K}_{j\_s})}{\partial t_i} \cdot \mathbf{T}_{j\_LG} \cdot \mathbf{T}_{j\_01} \right) \cdot \mathbf{u}_G \right] . \quad (35)$$

The relation $\frac{\partial \mathbf{f}}{\partial t_i}$ is often zero and the derivative of the all the element components of the stiffness matrix can be realized as follows [6]

$$\frac{\partial (\mathbf{K}_{j\_m} + \mathbf{K}_{j\_b} + \mathbf{K}_{j\_s})}{\partial t_i} = \frac{\delta_{ij}}{t_i} \cdot (\mathbf{K}_{j\_m} + 3 \cdot \mathbf{K}_{j\_b} + \mathbf{K}_{j\_s}) . \quad (36)$$

The particular membrane, bending and shear matrices are presented in [1, 7].

Finally, the derivative of the von Mises stress (at the top and at the bottom surfaces) with respect to the element thickness t_i is the following

$$\frac{\partial \sigma^2_{j\_ekv}\big|_{top}}{\partial t_i} = \frac{1}{2\sigma_{j\_ekv}\big|_{top}} \cdot \frac{\partial \sigma^2_{j\_ekv}\big|_{top}}{\partial t_i} \quad \text{and}$$

$$\frac{\partial \sigma_{j\_ekv}\big|_{bot}}{\partial t_i} = \frac{1}{2\sigma_{j\_ekv}\big|_{bot}} \cdot \frac{\partial \sigma^2_{j\_ekv}\big|_{bot}}{\partial t_i} . \quad (37)$$

All the presented approaches have been implemented into Matlab's FE software MATFEM developed by the authors.

Example 2

Determine the element stress derivative (eqs. 32a, 32b) with respect to the thickness t_1 and t_2 of the shell structure in Fig. 4.

Let's consider the following input parameters: elasticity modulus $E = 3.106$ MPa, Poisson's ratio $\mu = 0.3$, thicknesses $t_1 = 3$ mm and $t_2 = 2$ mm and force $F_Z = 2500$ N concentrated into each node of the top curved surface.

Stress gradient values for the chosen elements – analytical vs. numerical calculation — Tab. 2

Nr. of element	Stress gradient with respect t_1		Nr. of element	Stress gradient with respect t_2	
	Analytically	Numerically		Analytically	Numerically
4	180.6925	180.8217	81	72.1432	72.1356
15	178.1929	178.3464	66	56.8617	56.8841
12	172.7673	172.9401	65	56.5136	56.5514
7	172.2105	172.3427	92	54.8065	54.8449
53	170.2041	170.4455	80	52.5394	52.5649

The chosen calculated values of the stress gradients are written in Table 2. The presented analytic stress gradient calculation was confronted with the classical numerical computational approach ($\Delta\sigma_j / \Delta t_i$). A graphic presentation of the stress gradients distribution in each of the elements is in Figs. 5 and 6.

The results document the influence of both parameters on the stresses and the major signification of thickness t_1. This information may be used for the next optimizing process.

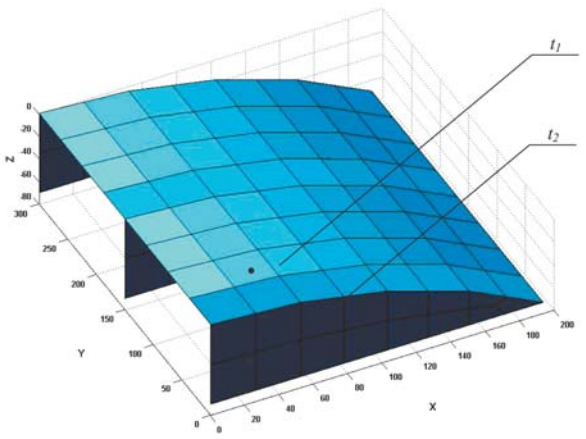


Fig. 4 Half model of the analysed shell structure in MATFEM

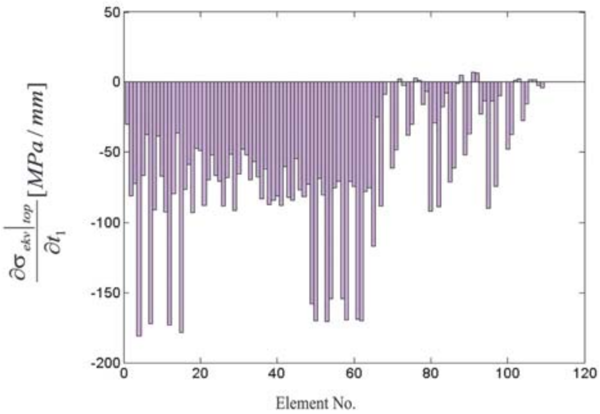


Fig. 5 Stress sensitivity with respect to t_1

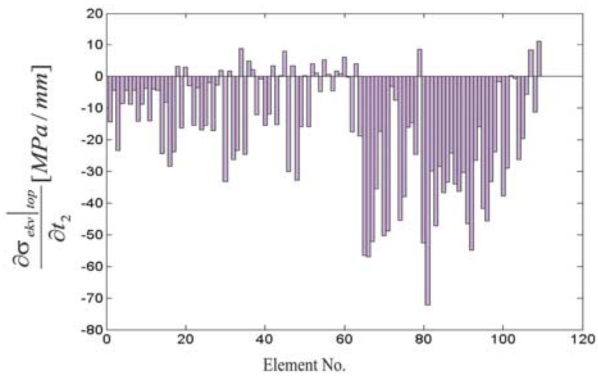


Fig. 6 Stress sensitivity with respect to t_2

4. Conclusion

The presented work deals with the theoretical aspects and numerical realization of the stress sensitivity analysis of the beam and shell finite elements focused on its cross section parameters (the area in the case of the beam element and the thickness in the case of the thin shell element). The whole computational procedure was inbuilt into Matlab's software module MATFEM. Testing examples support the authors' considerations about effectiveness of the proposed method.

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